

# Finitary Spacetime Sheaves of Quantum Causal Sets: Curving Quantum Causality

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## Abstract

A locally finite, causal and quantal substitute for a locally Minkowskian principal fiber bundle  $\mathcal{P}$  of modules of Cartan differential forms  $\Omega$  over a bounded region  $X$  of a curved  $C^\infty$ -smooth differential manifold spacetime  $M$  with structure group  $\mathbf{G}$  that of orthochronous Lorentz transformations  $L^+ := SO(1, 3)^\uparrow$ , is presented.  $\mathcal{P}$  is the structure on which classical Lorentzian gravity, regarded as a Yang-Mills type of gauge theory of a  $sl(2, \mathbb{C})$ -valued connection 1-form  $\mathcal{A}$ , is usually formulated. The mathematical structure employed to model this replacement of  $\mathcal{P}$  is a principal finitary spacetime sheaf  $\tilde{\mathcal{P}}_n$  of quantum causal sets  $\tilde{\Omega}_n$  with structure group  $\mathbf{G}_n$ , which is a finitary version of the group  $\mathbf{G}$  of local symmetries of General Relativity, and a finitary Lie algebra  $\mathfrak{g}_n$ -valued connection 1-form  $\mathcal{A}_n$  on it, which is a section of its sub-sheaf  $\tilde{\Omega}_n^1$ .  $\mathcal{A}_n$  is physically interpreted as the dynamical field of a locally finite quantum causality, while its associated curvature  $\mathcal{F}_n$ , as some sort of ‘finitary Lorentzian quantum gravity.

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...The locality principle seems to catch something fundamental about nature... Having learned that the world need not be Euclidean in the large, the next tenable position is that it must at least be Euclidean in the small, a manifold. The idea of infinitesimal locality presupposes that the world is a manifold. But the infinities of the manifold (the number of events per unit volume, for example) give rise to the terrible infinities of classical field theory and to the weaker but still pestilential ones of quantum field theory. The manifold postulate freezes local topological degrees of freedom which are numerous enough to account for all the degrees of freedom we actually observe.

The next bridgehead is a dynamical topology, in which even the local topological structure is not constant but variable. The problem of enumerating all topologies of infinitely many points is so absurdly unmanageable and unphysical that dynamical topology virtually forces us to a more atomistic conception of causality and space-time than the continuous manifold... (D. Finkelstein, 1991)

## 1. INTRODUCTION CUM PHYSICAL MOTIVATION

We are still in need of a cogent quantum theory of gravity. A quantum field theoretic scenario for General Relativity (GR) is assailed by non-renormalizable infinities coming from the singular values of fields that are assumed to propagate and interact on a smooth spacetime manifold. Most likely, it is our modeling of spacetime after a  $C^\infty$ -smooth differential manifold that is the culprit for this unpleasant situation. We can hardly expect Nature to have any infinities, but we can be almost certain that it is our own theoretical models of Her that are of limited applicability and validity.

The present paper takes a first step towards arriving at an operationally sound, locally finite, causal and quantal model of classical Lorentzian gravity from a finitary spacetime sheaf (finsheaf) theoretic point of view. Classical Lorentzian gravity is regarded as a Yang-Mills type of gauge theory of a  $sl(2, \mathbb{C})$ -valued connection 1-form  $\mathcal{A}$  that is suitably formulated on a locally Minkowskian principal fiber bundle  $\mathcal{P}$  of modules of Cartan differential forms  $\Omega$  over a bounded region  $X$  of a curved  $C^\infty$ -smooth differential manifold spacetime  $M$  with structure group  $\mathbf{G}$  that of orthochronous Lorentz transformations  $L^+ := SO(1, 3)^\uparrow$ . A principal finsheaf  $\vec{\mathcal{P}}_n$  of quantum causal sets (qausets<sup>1</sup>)  $\vec{\Omega}_n$  having as structure group a finitary version  $\mathbf{G}_n$  of  $L^+$ , together with a finitary spin-Lorentzian connection  $\mathcal{A}_n$  which is a  $\mathfrak{g}_n$ -valued section of the sub-sheaf  $\vec{\Omega}_n^1$  of reticular 1-forms of  $\vec{\mathcal{P}}_n$ , is suggested as a locally finite model, of strong operational character, of the dynamics of the quantum causal relations between events and their local causal symmetries in a bounded region  $X$

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<sup>1</sup>Since ‘causal sets’ are coined ‘causets’ for short by Sorkin (private communication), ‘quantum causal sets’ may be similarly nicknamed ‘qausets’.

of a curved smooth spacetime manifold  $M$ . In short, we propose  $(\vec{\mathcal{P}}_n, \mathcal{A}_n)$  as a finitary, causal and quantal replacement of the classical gravitational spacetime structure  $(\mathcal{P}, \mathcal{A})$ <sup>2</sup>. The theoretical model  $(\vec{\mathcal{P}}_n, \mathcal{A}_n)$  is supposed to be a preliminary step in yet another attempt at viewing the problem of ‘quantum gravity’ as the dynamics of a local, finitistic and quantal version of a variable causality<sup>3</sup> (Finkelstein, 1988, 1989, 1991, 1996, Bombelli *et al.*, 1987, Sorkin, 1990, 1995, Raptis, 2000*f*).

In more detail, the continuous (*ie*,  $C^0$ ) topology of a bounded region  $X$  of a spacetime manifold  $M$  has been successfully approximated by so-called ‘finitary topological spaces’ which are mathematically modeled after partially ordered sets (posets) (Sorkin, 1991). The success of such coarse approximations of the topological spacetime continuum rests on the fact that an inverse system consisting of finer-and-finer finitary posets possesses, at the maximum (finest) resolution of  $X$  into its point-events, a limit space that is effectively homeomorphic to  $X$  (Sorkin, 1991).

In a similar way, coarse approximations of the continuous (*ie*,  $C^0$ ) spacetime observables on  $X$  have been soundly modeled after so-called ‘finitary spacetime sheaves’ (finsheaves) which are structures consisting of continuous functions on  $X$  that are locally homeomorphic to the finitary posets of Sorkin (Raptis, 2000*a*). Also, an inverse system of such finsheaves was seen to ‘converge’, again at maximum refinement and localization of  $X$  to its point-events, to  $S(X)$ -the sheaf of (germs of sections of) continuous spacetime observables on  $X$  (Raptis, 2000*a*).

In (Raptis and Zapatrin, 2000), an algebraic quantization procedure of Sorkin’s finitary poset substitutes for continuous spacetime topology was presented, first by associating with every such poset  $P$  a non-commutative Rota incidence algebra  $\Omega(P)$ , then by quantally interpreting the latter’s structure. The aforementioned limit of a net of such quantal incidence algebras was interpreted as Bohr’s correspondence principle in the sense that the continuous spacetime topology emerges, as a classical structure, from some sort of decoherence of the underlying discrete and coherently superposing quantum Rota-algebraic topological substrata (Raptis and Zapatrin, 2000). The operationally pragmatic significance of the latter, in contradistinction to the ideal and, because of it, pathological<sup>4</sup> event structure that the classical differential manifold

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<sup>2</sup>Our scheme may be loosely coined a ‘finitary Lorentzian quantum gravity’, although it is more precise to think of  $\vec{\mathcal{P}}_n$  as a finitary, causal and quantal substitute for the structure  $\mathcal{P}$  on which GR is cast as a gauge theory, rather than directly of GR on it *per se*. For instance, we will go as far as to define curvature  $\mathcal{F}_n$  on  $\vec{\mathcal{P}}_n$ , but we will not give an explicit expression of the Einstein equations on it. The latter is postponed to another paper (Raptis, 2000*f*).

<sup>3</sup>See opening quotation above.

<sup>4</sup>Due to the unphysical infinities in the form of singularities from which the classical and quantum field theories, which are defined on the operationally ideal spacetime continuum, suffer (see also

model of spacetime stands for, was also emphasized by Raptis and Zapatin.

Furthermore, it has been argued (Raptis and Zapatin, 2000) that, in view of the fact that the  $\Omega(P)$ s were seen to be discrete differential manifolds in the sense of Dimakis and Müller-Hoissen (1999), not only the continuous ( $C^0$ ) topological, but also the smooth (*ie*,  $C^\infty$ ) differential structure of classical spacetime, emerges at the operationally ideal classical limit of finest resolution of a net of quantal incidence algebras. Since only at this ideal classical limit of an inverse system of such quantum topological substrata the local structure of the differential spacetime manifold emerges<sup>5</sup>, the substrata were conceived as being essentially alocal structures (Raptis and Zapatin, 2000), with this ‘a-locality’ signifying some sort of independence of these algebraic structures from the classical conception of spacetime as a smooth background geometric base space. Similarly, the finsheaf theoretic approach developed in (Raptis, 2000a), with its finitary algebraic-operational character, strongly emphasizes the physical significance of such a non-commitment to an inert background geometrical base spacetime manifold, as well as its accordance with the general operational, ultimately pragmatic, philosophy of quantum theory (Finkelstein, 1996).

Moreover, at the end of (Raptis, 2000a), it is explicitly mentioned that by assuming further algebraic structure for the stalks of the aforementioned finsheaves, as for instance by considering sheaves of incidence algebras over Sorkin’s finitary topological posets, at the limit of maximum resolution of a net of such finsheaves of Rota algebras, which can also be regarded as Bohr’s classical limit *à la* Raptis and Zapatin (2000), the differential triad  $(X, \Omega := \oplus_i \Omega^i, D)$  should emerge. The latter stands for the sheaf of modules of Cartan differential forms  $\Omega$  on the smooth  $X$ , equipped with the Kähler-Cartan differential operator  $D$  which effects (sub)sheaf morphisms of the following sort  $D : (X, \Omega^i) \rightarrow (X, \Omega^{i+1})$  (Mallios, 1998). Thus, a finsheaf of Rota incidence algebras is expected to be a sound model of locally finite, as well as quantal, ‘approximations’ of the smooth spacetime observables-the classical spacetime dynamical fields<sup>6</sup>.

Parenthetically, and with an eye towards the physical interpretation to be given opening quotation).

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<sup>5</sup>That is to say, the spacetime event and the space of covariant directions tangent to it (*ie*, its cotangent space of differential forms).

<sup>6</sup>We tacitly assume that the classical model for spacetime and the fields inhabiting, propagating and interacting on it is that of a 4-dimensional differential (or  $C^\infty$ -smooth) manifold, with fiber-spaces  $\Omega^n$  of smooth Cartan exterior  $n$ -forms attached (in fact,  $0 \leq n \leq 4$ , but we will not be further concerned about questions of dimensionality; see discussion in (c) of section 6). Physical fields are then modeled after cross-sections of this Cartan fiber bundle  $\mathcal{P}$  of smooth exterior forms (Göckeler and Schücker, 1990, Baez and Muniain, 1994).

subsequently to our mathematical model, we should mention at this point that the inverted commas over the word ‘approximations’ in the last sentence above may be explained as follows: after the successful algebraic quantization of Sorkin’s discretized spacetimes in (Raptis and Zapatrin, 2000), it has become clear that the resulting alocal quantum topological incidence algebras  $\Omega(P)$  associated with the finitary topological posets  $P$  in (Sorkin, 1991) should not be thought of as approximations-proper of the classical smooth differential forms like their corresponding  $P$ s or the finsheaves  $S_n$  in (Raptis, 2000a) approximate the  $C^0$ -topological manifold structure of classical spacetime, as if a geometric spacetime exists as a background base space ‘out there’. Rather, they should be regarded as operationally pragmatic and relatively autonomous quantum spacetime structures an inverse system<sup>7</sup> of which possesses an operationally ideal (*ie*, unobservable in actual experiments) and classical, in the sense of Bohr, limit structure isomorphic to the differential manifold model of spacetime (Raptis and Zapatrin, 2000). From this viewpoint, the quantum topological incidence algebras  $\Omega(P)$  (and their qauset relatives in (Raptis, 2000b)) are regarded as being physically fundamental (primary) and their correspondence limit geometric manifold structure as being derivative (secondary), ultimately, their emergent classical counterpart in much the same way that the classical Poisson algebra of observables on the geometric phase (state) space of a classical mechanical system is the ‘classical decoherence limit’<sup>8</sup> or the ‘classical undeformation’<sup>9</sup> of the Heisenberg algebra of an underlying quantum system in the operationally ideal situation of non-interfering quantally and non-perturbing dynamically<sup>10</sup> operations of observation (determination) of the properties of the latter<sup>11</sup>. Properly conceived, it is the classical theory (model) that should be thought of as an approximation of the deeper quantum theory (model), not the other way around (Finkelstein, 1996). Thus, ‘quantum replacements’ or ‘quantum substitutes’ instead of ‘approximations’ will be used more often from now on to describe our finsheaves (of qausets), although they were initially conceived as approximations-proper of the continuous spacetime topology in (Raptis, 2000a) as it was originally motivated by Sorkin (1991). In total, this non-acceptance of ours of spacetime as an inactive smooth geometric receptacle of the physical fields or as a background stage that supports their dynamical propagation, that is passively existing as a static state space ‘out there’ and whose structure is fixed *a priori* independently of our experimental actions on or operations of observation of ‘it’, is the essence of the operationally sound quantum

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<sup>7</sup>‘Collection’ or even ‘ensemble’ are also appropriate synonyms to ‘system’ in this context.

<sup>8</sup>That is to say, when coherent quantum superpositions between observables are lifted.

<sup>9</sup>That is to say, the undoing of the formal procedure of ‘quantum deformation’.

<sup>10</sup>That is to say, infinitely smooth.

<sup>11</sup>Altogether, formally ‘as  $\hbar \rightarrow 0$ ’.

physical semantics that we will give to our algebraic finsheaf model in the present paper.

In GR, the classical theory of gravity which is based on the structural assumption that spacetime is a 4-dimensional pseudo-Riemannian manifold  $M$ , the main dynamical variable is the smooth Lorentzian spacetime metric  $g_{\mu\nu}$  which is physically interpreted as the gravitational potential. The local relativity group of GR, in its original formulation in terms of the Lorentzian metric  $g_{\mu\nu}$ , is the orthochronous Lorentz group  $L^+ := SO(1, 3)^\uparrow$ . GR may also be formulated in terms of differential forms on the locally Minkowskian bundle  $\mathcal{P}$  (Göckeler and Schücker, 1990)<sup>12</sup>. Equivalently, in its gauge theoretic spinorial formulation (Bergmann, 1957<sup>13</sup>, Baez and Muniain, 1994<sup>14</sup> type of gauge theory of a  $sl(2, \mathbb{C})$ -valued 1-form  $\mathcal{A}$ -the spin-Lorentzian connection field, which represents the gravitational gauge potential. A sound model for this theory is a principal fiber bundle  $\mathcal{P}$  over (the region  $X$  of) the  $C^\infty$ -smooth spacetime manifold  $M$ , with structure group  $\mathbf{G} = SL(2, \mathbb{C})$ <sup>15</sup> and a non-flat connection 1-form  $\mathcal{A}$  taking values in the Lie algebra  $\mathfrak{g} = sl(2, \mathbb{C})$  of  $\mathbf{G}$ <sup>16</sup>, totally,  $(X, \mathcal{P}, \mathbf{G}, \mathcal{A})$ <sup>17</sup>. Thus,

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<sup>12</sup>See chapter on Einstein-Cartan theory. We call  $\mathcal{P}$  ‘the Cartan principal fiber bundle with structure group the orthochronous Lorentz group  $L^+$  of local invariances of GR’. See sections 2 and 5.

<sup>13</sup>In this theory,  $g_{\mu\nu}$  is replaced by a field of four  $2 \times 2$  Pauli spin-matrices.

<sup>14</sup>We refer to Ashtekar’s modification of the Palatini formulation of GR by using new spin variables (Ashtekar, 1986). In this theory, only the self-dual part  $\mathcal{A}^+$  of a spin-Lorentzian connection  $\mathcal{A}$  is regarded as being physically significant. In (Raptis, 2000*e*) this is used as an example to argue that the fundamental quantum time asymmetry expected of ‘the true quantum gravity’ (Penrose, 1987) is already built into the kinematical structure of a locally finite, causal and quantal version of that theory modeled after curved finitary spacetime sheaves (or schemes) of qausets.

<sup>15</sup>A principal fiber bundle with structure group  $\mathbf{G}$  may also be called a ‘ $\mathbf{G}$ -bundle’ for short.

<sup>16</sup>Since locally in the group fiber (*ie*, Lie algebra-wise in the fiber space) of the  $\mathbf{G}$ -bundle  $\mathcal{P}$   $sl(2, \mathbb{C})$  is isomorphic to the Lie algebra  $\ell^+ = so(1, 3)^\uparrow$  of the orthochronous Lorentz group  $L^+$ ,  $\mathcal{P}$  may equivalently be thought of as having the latter as structure group  $\mathbf{G}$ . Due to this local isomorphism,  $\mathcal{A}$  is given the epithet ‘spin-Lorentzian’ and the same symbol  $\mathcal{P}$  is used above for both the Cartan (Lorentzian) and the Bergmann (spin)  $\mathbf{G}$ -bundles. Thus,  $\mathcal{P}$  is called ‘the Cartan-Bergmann  $\mathbf{G}$ -bundle’. See section 5 for more on this local isomorphism between the Cartan and the Bergmann  $\mathbf{G}$ -bundles.

<sup>17</sup>The name ‘principal’ is usually reserved only for the group  $\mathbf{G}$ -bundle or sheaf, while the vector or algebra sheaf that carries it, in our case  $\Omega$ , is called ‘associated’ (Mallios, 1998). Here we use one symbol,  $\mathcal{P}$ , and one name, ‘principal’, for both the  $\mathbf{G}$ -sheaf of orthochronous Lorentz transformations  $L^+$  and its associated locally Minkowskian sheaf of differential forms  $\Omega$ . Conversely, in section 4 we first define  $\Omega$  as an algebra sheaf and then we coin the  $\mathbf{G}$ -sheaf of its Lorentz symmetries ‘adjoint’. There is no misunderstanding:  $\Omega$  is associated with  $\mathbf{G}$ , or vice versa,  $\mathbf{G}$  is adjoint to  $\Omega$ , and together they constitute the principal sheaf  $\mathcal{P}$ . Nevertheless, we apologize to the mathematical purist for this slight change in nomenclature.

by the discussion in the penultimate paragraph, it follows that a principal finsheaf of quantum incidence algebras, together with a non-flat connection taking values in their local symmetries, may be employed to model a locally finite and quantal version of Lorentzian gravity in its gauge theoretic formulation on a smooth spacetime manifold.

However, there seem to be *a priori* two serious problems with such a model. On the one hand, only Riemannian (*ie*, positive definite) metric connections may be ‘naturally’ defined on discrete differential manifolds such as our Rota incidence algebras (Dimakis and Müller-Hoissen, 1999), and on the other, the anticipated classical limit sheaf or fiber bundle  $(X, \Omega, D)$  is flat (Mallios, 1998)<sup>18</sup>. The first comes into conflict with the indefinite character of the local spacetime metric of GR<sup>19</sup>, thus also with its local relativity group<sup>20</sup>; while the second, with the general relativistic conception of the gravitational field strength as the non-vanishing curvature of spacetime.

One should not be discouraged, for there seems to be a way out of this double impasse which essentially motivated us to consider finsheaves of qausets in the first place. First, to deal with the ‘signature problem’, we must change physical interpretation for the algebraic structure of the stalks of the aforementioned finsheaf of quantal incidence algebras from ‘topological’ to ‘causal’. This means that we should consider finsheaves of the qausets in (Raptis, 2000*b*), rather than finsheaves of the quantum discretized spacetime topologies in (Raptis and Zapatrin, 2000). Indeed, Sorkin (1995), in the context of constructing a plausible theoretical model for quantum gravity, convincingly argues for a physical interpretation of finitary posets as locally finite causets (Bombelli *et al.*, 1987, Sorkin, 1990) and against their interpretation as finite topological spaces or simplicial complexes (Alexandrov, 1956, 1961, Raptis and Zapatrin, 2000). Similar arguments against a non-relativistic, spatial conception of topology and for a temporal or causal one which is also algebraically modeled with a quantum interpretation given to this algebraic structure, like the quantum causal topology of the qausets in (Raptis, 2000*b*), are presented in (Finkelstein, 1988). Ancestors of the causet idea are the classic works of Robb (1914), Alexandrov (1956, 1967) and Zeeman (1964, 1967) which show that the entire topology and conformal geometry of Minkowski space  $\mathcal{M}$ , as well

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<sup>18</sup>Dimakis and Müller-Hoissen (1999) also mention the fact that the (torsionless) Riemannian metric connection  $\nabla$  of the universal differential calculus on a discrete differential manifold is flat, in that it reduces to the nilpotent Kähler-Cartan differential  $D$  whose curvature  $\mathcal{R}$  is zero, since  $\mathcal{R} := \nabla^2 = D^2 = 0$ .

<sup>19</sup>In GR, the local metric field  $g_{\mu\nu}$  is Lorentzian (of signature 2), not Euclidean (of trace 4).

<sup>20</sup>The group of local isometries of GR, at least in its spinorial gauge theoretic formulation mentioned above, is taken to be  $SL(2, \mathbb{C})$ -the double cover of the orthochronous Lorentz group  $L^+ = SO(1, 3)^\uparrow$  of local invariances of  $g_{\mu\nu}$  that also locally preserve the orientation of time, not the 4-dimensional unimodular Euclidean rotations in  $SO(4)$ . In this sense GR is a theory of (locally) Lorentzian gravity (see next section and 5).

as its relativity group  $L^+$  of global orthochronous Lorentz transformations modulo spacetime volume-preserving maps, can be determined by modeling the causal relation between its events after a partial order. Alternatively, the spirit of deriving the entire geometry of the Lorentzian spacetime manifold from causality modeled after a partial order, is captured by the following words taken from (Bombelli *et al.*, 1987)<sup>21</sup>:

There is a fact, insufficiently appreciated in our view, that a classical space-time's causal structure comes very close to determining its entire geometry. By the causal structure of a space-time, one means the relation  $P$  specifying which events lie to the future of which other events. Ordinarily, one thinks of space-time as a topological manifold  $M$ , endowed with a differentiable structure  $S$ , with respect to which a metric  $g_{ab}$  is defined. Then the causal order  $P$  is regarded as derived from the lightcones of  $g$ . However, one can also go the other way: Given a space-time obeying suitable smoothness and causality conditions (and of dimensionality  $> 2$ ), let us retain from all its structure the information embodied in the order relation  $P$ . Then we can recover from  $P$  not only the topology of  $M$ , but also its differentiable structure, and the conformal metric,  $g_{ab}/|\det(g)|^{1/n}$ . Now a partial ordering is a very simple thing, and it is natural to guess that in reality  $g_{ab}$  should be derived from  $P$  rather than the other way around...

On the other hand, causality as a partial order, while it solves the ‘signature problem’, is unable to adequately address the second ‘curvature problem’ mentioned above, since it determines the Minkowski space  $\mathcal{M}$  of Special Relativity (and its Lorentz symmetries) which is flat (and its Lorentz symmetries are global). Our way out of this second ‘curvature impasse’ involves a rather straightforward localization or gauging of the qausets in (Raptis, 2000b), by considering a non-flat connection on a finsheaf of such quantally and causally interpreted incidence algebras, thus by emulating the work of Mallios (1998)<sup>22</sup> that studies Yang-Mills gauge connections on  $\mathbf{G}$ -sheaves of vector spaces and algebras in general. This gauging of quantum causality translates in a finitary and quantal setting the fact that the classical theory of gravity, GR, may be regarded as Special Relativity (SR)-localized or being gauged<sup>23</sup>. This connection variable is supposed to represent the dynamics of an atomistic local quantum causality as the latter is algebraically encoded stalk-wise in the finsheaf (*ie*, in the qausets). The result may be regarded as the first essential step towards formulating a finitary dynamical scenario for the qauset stalks of the sheaf which, in turn, may be

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<sup>21</sup>See also (Bombelli and Meyer, 1989, Sorkin, 1990).

<sup>22</sup>Albeit, in a finitary causal and quantal context.

<sup>23</sup>So that the spacetime metric, or its associated connection, become dynamical field variables (Torretti, 1981).



physically interpreted as a finitary and causal model of the still incompletely or even well-formulated Lorentzian quantum gravity. Equivalently, and in view of the sound operational interpretation given to the topological incidence algebras in (Raptis and Zapatin, 2000) as well as to the topological finsheaves in (Raptis, 2000a), our model may be physically interpreted as locally finite and quantal replacements of the dynamics of the local causal relations between events and their local causal symmetries<sup>24</sup>, in a limited (finite or bounded), by our own domain of experimental activity (*ie*, laboratory) (Raptis and Zapatin, 2000), region  $X$  of the smooth spacetime manifold  $M$ <sup>25</sup>. As we mentioned above, the latter ‘exists’ only as a ‘surrogate background space’ that helps one remember where the discreteness of our model comes from, but it is not essential to the physical problem in focus. The spacetime continuum, as a ‘base space’, is only a geometrical scaffolding that supports our structures<sup>26</sup>, but that should also be discarded after their essentially alocal-algebraic, quantal-operational and causal (*ie*, non-spatial, but temporal) nature is explicated and used for our problem in focus. Then, the aforementioned correspondence principle for quantal topological incidence algebras may be used on (an inverse system of) the principal finsheaves of qausets and their non-flat spin-Lorentzian connections in order to recover the classical spacetime structure on which GR is formulated, as the classical theory of gravity, at the classical and operationally ideal limit of resolution (*ie*, of infinite localization and infinitesimal/differential separation) of spacetime into its events. This classical limit spacetime model for GR, as a gauge theory, is the one mentioned above, namely, a principal fiber bundle  $\mathcal{P}$  of modules of smooth Cartan differential forms  $\Omega$ , over (a region  $X$  of) a  $C^\infty$ -smooth Lorentzian spacetime manifold  $M$ , with structure group  $\mathbf{G} = SL(2, \mathbb{C})$  or its locally isomorphic  $SO(1, 3)^\dagger$ , and a non-flat  $sl(2, \mathbb{C})$ -valued gravitational gauge connection 1-form  $\mathcal{A}$  on it<sup>27</sup>.

The present paper is organized as follows: in the next section we propose and discuss in detail finitary versions of the principles of Equivalence and Locality of GR, as well as of their ‘corollaries’, the principles of Local Relativity and Local Superposition, that are expected to be operative at the locally finite setting that we place our first step at modeling ‘finitary Lorentzian quantum gravity’ after ‘curving quantum causality by

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<sup>24</sup>That is to say, the dynamics of local quantum causality or ‘local quantum causal topology’ and its symmetries.

<sup>25</sup>See sections 5 and 6.

<sup>26</sup>In the sense that ‘it avails itself to us as a topological space’ by providing sufficient (but not necessary !) conditions for the definition of  $\mathcal{A}_n$  which is the main dynamical variable in our theoretical scheme. See section 5.

<sup>27</sup>Thus,  $\mathcal{A}$  is a cross-section of the  $\Omega^1$  sub-bundle of the Cartan-Bergmann  $\mathbf{G}$ -bundle  $\mathcal{P}$  taking values in  $sl(2, \mathbb{C}) \simeq \ell^+$ .

gauging a principal finsheaf of qausets'<sup>28</sup>. In section 3 we review the algebraic model of flat quantum causality proposed in (Raptis, 2000*b*), namely, the qauset, and emphasize its local aspects to be subsequently gauged (in section 5). In section 4 we recall the topological finsheaves from (Raptis, 2000*a*), then we define finsheaves of qausets and their local symmetries. At the end of the section, a sound operational interpretation of finsheaves of qausets and their symmetries is given, so that our theory is shown to have a strong philosophical support as well. In section 5 we suggest that for localizing or gauging and, as a result, curving quantum causality, a principal finsheaf of qausets having as structure group a finitary version of  $SO(1,3)^\dagger$ , together with a discrete and local sort of a non-flat, spin-Lorentzian connection  $\mathcal{A}_n$  on it, is an operationally sound model.  $\mathcal{A}_n$  is then physically interpreted as a finitary, local, causal and quantal topological variable whose non-zero curvature stands for a finitary, causal and quantal model of Lorentzian gravity. We conclude the paper by discussing the soundness of this finsheaf model of finitary and causal Lorentzian quantum gravity as well as six physico-mathematical issues that derive from it.

## 2. PHYSICAL PRINCIPLES FOR FINITARY LORENTZIAN QUANTUM GRAVITY

In this section we commence our endeavor to model connection (and its associated curvature) in a curved finitary quantum causal setting by establishing heuristic physical principles that must be encoded in the very structure of our mathematical model<sup>29</sup> on which the dynamics of a locally finite quantum causality is going to be founded in section 5. The four physical principles to be suggested here will be seen to be the finitary and (quantum) causal analogues of the ones of Equivalence, Locality, as well as their ‘corollary’ principles of Local Relativity and Local Superposition respectively, of GR which is formulated as a gauge theory in  $\mathcal{P}$  over a differential manifold space-time  $M$ . We have chosen these principles from the theory of classical gravity, because they show precisely in what way the latter is a type of gauge theory, and also because they will prepare the reader for our localization or gauging and curving of qausets in section 5.

The first physical principle from GR that we would like to adopt in our inherently reticular scenario, so that curvature may be naturally implemented and straightforwardly interpreted as gravity in a finitary (quantum) causal context like ours, is that

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<sup>28</sup>The word ‘gauging’ pertains to the aforementioned implementation of a non-flat gauge connection  $\mathcal{A}_n$  on the finsheaf in focus.

<sup>29</sup>The ‘principal finsheaf of qausets with a non-flat finitary spin-Lorentzian connection on it’, to be built progressively in the next three sections.

of equivalence (EP). We borrow from GR the following intuitively clear version of the EP:

*Classical Equivalence Principle (CEP)*: the curved spacetime of GR is locally Minkowskian; thence flat. That is to say, the space tangent to every spacetime event is isomorphic to flat Minkowski space  $\mathcal{M}$ . As we mentioned in the introduction, in this sense GR may be thought of as SR made local or been point-wise (event-wise) gauged. Expressed thus, the CEP effectively encodes Einstein’s fundamental insight that locally the gravitational field  $g_{\mu\nu}$  can be ‘gauged away’ or be reduced to the constant and flat Minkowski metric  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$  of SR<sup>30</sup> by passing to a locally inertial frame<sup>31</sup> (Torretti, 1981).

What is important to emphasize in this formulation of the CEP is that in GR  $\mathcal{M}$  assumes a local kinematical role, in the sense that an isomorphic copy of it is erected, as some kind of ‘fiber space’, over each event of the differential manifold spacetime, so that every individual fiber is physically interpreted as ‘an independent (of the other  $\mathcal{M}$ -fibers) vertical world of spacetime possibilities along which the dynamically variable  $g_{\mu\nu}$  can be reduced to the constant  $\eta_{\mu\nu}$ ’<sup>32</sup>. It follows that the symmetries of gravity are the isometries of  $\mathcal{M}$ -localized; hence, one arrives at a gauged or localized version of the Lorentz group as the invariances of GR. This motivates us to formulate the Classical Local Relativity Principle (CLRP) which, in a sense, is a corollary of the CEP above:

*Classical Local Relativity Principle (CLRP)*: The group of local (gauge) invariances of GR is isomorphic to the orthochronous Lorentz group  $L^+ = SO(1, 3)^\uparrow$  of symmetries of the Minkowski space of SR.

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<sup>30</sup>Since  $\eta_{\mu\nu}(x)$  delimits the Minkowski lightcone at  $x$  for every  $x \in M$ , which, in turn, defines the local causal relations between events in the Minkowski space tangent to  $x$ , the gravitational potential  $g_{\mu\nu}$  may be alternatively interpreted as ‘the dynamical field of local causality’.

<sup>31</sup>Equivalently put, CEP states that a body gravitated under a constant gravitational field intensity  $\Gamma$  is physically indistinguishable from a uniformly accelerated one with constant acceleration  $\gamma = \Gamma$ , a statement that entails the local equivalence between the body’s gravitational and inertial mass.

<sup>32</sup>The epithet ‘kinematical’ for CEP may be also justified as follows: for every point-event  $x$  of the pseudo-Riemannian spacetime manifold  $M$  of GR, there is a coordinate system among all the possible general coordinate frames, a so-called locally inertial one (the epithets ‘normal’ or ‘geodesic’ may also be used instead of ‘inertial’), with respect to which  $g_{\mu\nu} = \eta_{\mu\nu}$  and  $(\frac{\partial g_{\mu\nu}}{\partial x_\lambda}) = 0$ , but exactly due to gravity, the second partial derivatives of the metric cannot be made to vanish and it is precisely the latter that constitute the curvature tensor  $R$  at  $x$ . Since the CEP involves the metric and its first derivatives, while its second derivatives acquire a dynamical interpretation as the force-field of gravity, one may regard CEP as a kinematical principle for the basic gravitational potential variable  $g_{\mu\nu}$  expressing a ‘(gauge) possibility for local flatness’ in GR by suitably choosing the local frame (gauge) at  $x$  to be inertial.

In sum, the curved spacetime of GR may be modeled after the locally Minkowskian tangent vector bundle  $TM := \bigcup_{x \in X \subset M} \mathcal{M}_x$ , which is a sub-bundle of the dual of the  $\mathbf{G}$ -bundle  $\mathcal{P}$  mentioned in the introduction that consists of modules of Cartan differentials<sup>33</sup> and has as structure group  $\mathbf{G} = SO(1,3)^\uparrow$ , together with a non-flat  $\mathfrak{g} = so(1,3)^\uparrow \simeq sl(2, \mathbb{C})$ -valued spin-Lorentzian  $\Omega^1$ -section  $\mathcal{A}$ .

Since, as it was mentioned in the introduction, causets effectively encode the geometry of flat Minkowski space  $\mathcal{M}$ , they can be thought of as local kinematical structures representing the possible local causal relations in an otherwise curved spacetime of events. The CEP, modified to fit a finitary, curved and causal situation like ours, reads:

*Finitary Equivalence Principle (FEP)*: a locally finite curved causal space is locally<sup>34</sup> a causet. Presumably, it is the transitivity of causality as a partial order that must be renounced due to gravity (Finkelstein, 1988, Raptis, 1998, 2000*b,c*)<sup>35</sup>.

In other words, a curved smooth spacetime, as a causal space, is not globally transitive; it is only locally (and kinematically) so<sup>36</sup>. Thus, the CEP may be restated as a correspondence or reduction principle: as the dynamically variable gravitational potential  $g_{\mu\nu}$  reduces locally to the constant  $\eta_{\mu\nu}$  in GR, causality becomes locally<sup>37</sup> the constant transitive partial order  $\rightarrow$ <sup>38</sup>. Equivalently, a curved finitary causal space, one having a causal relation not fixed to a globally transitive partial order, but with a dynamically variable local causality between its events, is only locally reducible to a transitive, flat ‘inertial causet’. Thus, as  $\mathcal{M}$  may be thought of as vertically extending, as an independent kinematical fiber space, over every event of the curved smooth spacetime manifold of GR, so an independent causet may be thought of as being raised over every point-event of a curved finitary causal space. Hence, the FEP almost mandates that a curved finitary causal space be modeled after a finsheaf (or a bundle) of causets (over a finitary spacetime). As a matter of fact, and also due to the finitary principle of Locality that we will formulate shortly, we will see that a curved finitary causal space should be modeled after a finsheaf of qausets (not of

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<sup>33</sup>That is to say, Minkowskian covariant-tangent/cotangent vectors which are dual to the Minkowskian contravariant vectors in the fibers  $\mathcal{M}_x$  of  $TM$ . See also section 5.

<sup>34</sup>Locality pending definition in our finitary context.

<sup>35</sup>Intuitively, gravity tilts the lightcone soldered at (*ie*, with origin) each event, thus renders causality an intransitive relation between them.

<sup>36</sup>That is to say, in the vertical direction along each of the Minkowskian fibers of the curved co-vector bundle  $\mathcal{P}$  above.

<sup>37</sup>As it was mentioned earlier, locality pending definition in our finitary scenario (see the principle of Finitary Locality below).

<sup>38</sup>With the CEP in mind, we may call ‘ $\rightarrow$ ’ ‘the inertial Minkowskian causality’. In a curved causal space causality only locally can be  $\rightarrow$  (CEP).

transitive causets) for discrete locality's sake. Thus, some kind of 'quantumness' will inevitably be infused into our model of the dynamics of finitary causality *ab initio*<sup>39</sup>. Before we give the Finitary Locality Principle and its 'corollary', the Finitary Local Superposition Principle, we give the finitary analogue of the CLRP above:

*Finitary Local Relativity Principle (FLRP)*: The local invariance structure of a curved finitary causal space is a finitary version of  $L^+$ . In a causal context, the work of Zeeman (1964, 1967) has shown that the symmetry structure  $L^+$  of the flat Minkowski continuum  $\mathcal{M}$ , regarded as a causal space with a causality relation between its events modeled after a (globally) inertial partial order  $\rightarrow$  which, in turn, derives from  $\mathcal{M}$ 's  $\eta_{\mu\nu}$ <sup>40</sup>, is isomorphic to the group  $\mathbf{G}$  of causal automorphisms of  $\mathcal{M}$ <sup>41</sup>. In our case, and in view of the FEP, we infer that a finitary version of  $L^+$ , which we call  $\mathbf{G}_n$ , comprises the local relativity structure group of a curved finitary causal space<sup>42</sup>. Now due to the local isomorphism mentioned in the introduction between the Lie algebras  $\ell^+ = so(1, 3)^\dagger$  and  $sl(2, \mathbb{C})$  in the smooth  $\mathbf{G}$ -bundle  $\mathcal{P}$ , we may alternatively say that  $\mathbf{G}_n$  is the finitary version of the local relativity group  $SL(2, \mathbb{C})$  of GR in its spinorial gauge theoretic formulation (Bergmann, 1957, Ashtekar, 1986, Baez and Muniain, 1994). A similar local relativity group for a curved finitary quantum causal space was proposed in (Finkelstein, 1988)<sup>43</sup> and Selesnick (1994) found that  $SL(2, \mathbb{C})$  is the local relativity group for Finkelstein's reticular and curved quantum causal net.

In all the principles and remarks above, we mentioned the word 'local' without having transcribed the notion of classical locality to a curved finitary causal scenario like ours. We do this now. The Classical Locality Principle (CLP) in GR may be summed up to the following assumption:

*Classical Locality Principle (CLP)*: The spacetime of GR is modeled after a differen-

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<sup>39</sup>This infusion converts our scheme to 'a model of the dynamics of finitary *quantum* causality'. See our formulation of the Finitary Local Superposition Principle below and the relevant discussion in section 6.

<sup>40</sup>See section 5 for more on this.

<sup>41</sup>Since the Alexandrov causal topology of  $\mathcal{M}$  is defined by  $\rightarrow$  (Alexandrov, 1956, 1967), it follows that  $\mathbf{G}$  is the group of causal homeomorphisms of  $\mathcal{M}$  (Torretti, 1981).

<sup>42</sup>Then,  $\mathbf{G}_n$  consists of the local causal homeomorphisms of the dynamically variable (and quantal) causal topology of a curved finitary causal space. Since we plan to model the latter after a finsheaf of qausets whose local structure is (by definition) the qauset stalks over this curved finitary causal base space,  $\mathbf{G}_n$  may be equivalently thought of as consisting of the group of homomorphisms (or automorphisms) of the quantal and causal incidence algebras representing these qauset stalks (Raptis, 2000b). (See also remarks at the end of this section on the significance of our choice to model the dynamics of a finitary quantum causality by a finsheaf of qausets.)

<sup>43</sup>We refer to the local  $SL_2$ -invariances of the dyadic cell there.

tial ( $C^\infty$ -smooth) manifold  $M$  (Einstein, 1924)<sup>44</sup>.

Since a locally finite causal model like ours does not involve (by definition) a continuous infinity of events like the  $M$  above, the CLP on  $M$  may be translated in finitary causal terms to the following requirement:

Definition of Finitary Locality (DFL): In a causet, locality pertains to physical properties, to be interpreted as observables or dynamical physical variables, with ‘effective range of action or dynamical variation’ restricted to empty Alexandrov sets<sup>45</sup>. Hence, we shall demand that the following physical principle be obeyed by our model of a curved finitary causal space:

Finitary Locality Principle (FLP): Dynamical relations on a causet  $(X, \rightarrow)$  involve only finitary local observables<sup>46</sup>.

Some scholia on DFL and FLP are due here. Since in our reticular scheme we can assume no dynamical properties varying between infinitesimally (*ie*, smoothly) separated events, we may as well define local physical observables as the entities that vary between nearest neighboring events called ‘contiguous’ from now on<sup>47</sup>. The FLP can be coined ‘the principle of contiguity in a finitary causal space’ and it is the reticular analogue of the CLP of GR, which, in turn, as it was posited above, may be summarized to the assumption of a 4-dimensional differential manifold model for spacetime (Einstein, 1924)<sup>48</sup>. Also, by the FEP above, we expect that in a curved finitary causal space gravity ‘cuts-off’ the transitivity of causality as a partial order and

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<sup>44</sup>Thus, the CLP may be viewed as the requirement that all the dynamical laws of physics must be differential equations, or more intuitively, that local dynamical actions connect (influence) infinitesimally or ‘differentially’ separated events living in the tangent space at each event of the smooth spacetime continuum. It follows that the CLP requires physical observables or dynamical variables to be modeled after (sections of) smooth differential forms (in  $\mathcal{P}$ ), as mentioned in the introduction. Thus, by ‘the local structure of the curved spacetime manifold  $M$ ’ we mean ‘an event  $x$  and the space of directions (contravariant vectors) tangent to it’ (Raptis and Zapatrin, 2000). In the bundle  $\mathcal{P}$  this pertains to its, Minkowskian by the CEP, fibers over each and every event  $x$  of its base space  $M$ .

<sup>45</sup>See (Bombelli *et al.*, 1987, Sorkin, 1990, Raptis, 2000b) and the next footnote for a definition of these.

<sup>46</sup>Thus, only dynamical changes of observables between ‘nearest neighboring events’ delimiting null Alexandrov sets in  $X$  (*ie*, ‘ $p, q \in X : (p \rightarrow q) \wedge (\nexists r : p \rightarrow r \rightarrow q)$ ’, or in terms of the Alexandrov interval bounded by  $p$  and  $q$ ,  $A(p, q) := \{r : p \rightarrow r \rightarrow q\} = \emptyset$ ) are regarded as being physically significant. This principle is an explication of the definition of non-mediated (immediate) physical dynamical actions in the DFL above. Thus, by the DFL we anticipate the gravitational connection  $\mathcal{A}$  in its finitary and causal version  $\mathcal{A}_n$ , which is supposed to be the main gravitational dynamical variable in our scheme, to be defined (as varying) on such immediate causal arrows. See section 5 for more on this.

<sup>47</sup>In the last footnote,  $p$  and  $q$  in the causet  $(X, \rightarrow)$  are contiguous.

<sup>48</sup>Parenthetically, we mention that in this paper Einstein concludes that the smooth geometrical

restricts the latter to empty Alexandrov causal neighborhoods of contiguous events.

At this point it must be mentioned that the FLP, apart from seeming rather natural to assume, was somehow ‘forced’ on us by discrete topological and local quantum causal considerations. In more detail<sup>49</sup>, it has been recently shown (Breslav *et al.*, 1999) that the generating relation  $\rho$  of the Rota topology of the incidence algebra  $\Omega$  associated with a poset finitary substitute  $P$  of a continuous spacetime manifold as in (Sorkin, 1991) is the same as the one generating the finite poset-topology of  $P$  if and only if one considers points in the Hasse diagram of the latter that are immediately connected by the partial order ‘ $\rightarrow$ ’ (*ie*, ‘contiguous events’). Then, if one interprets ‘ $\rightarrow$ ’ in the finitary poset causally instead of topologically as in (Sorkin, 1995, Raptis, 2000*b*), and gives a cogent quantum interpretation to the structure of the causal Rota algebra associated with it as in (Raptis and Zapatrin, 2000, Raptis, 2000*b*), one is led to infer that the physically significant, because local, causal connections between events in a qauset are the contiguous, immediate ones; hence the FLP above. This was first anticipated by Finkelstein (1988)<sup>50</sup>. The FLP promotes this conjecture to a ‘physical axiom’ (physical principle) concerning the finitary dynamics of local (quantum) causality in a curved locally finite causal space<sup>51</sup> in the same way that in the  $C^\infty$ -smooth spacetime  $M$  of GR locality was ‘forced’ on Einstein by  $M$ ’s own smoothness<sup>52</sup>.

According to the finitary principles formulated above, we may say that in the manifold model for spacetime, which is postulated up-front in GR for classical locality’s sake, may be thought of as an inert and absolute ‘aether’-like background structure on which the whole theory of GR and the mathematical language that supports it, classical Differential Geometry, is erected. In view of his characteristic dissatisfaction with any theory that employs structures that are absolute and non-dynamical, ultimately, ‘unobservable substances’, and in view of the reticular, molecular picture of Nature that the quantum revolution brought about, we infer that Einstein could not have been content with the smooth manifold model for spacetime (Einstein, 1936, 1956). This is a significant aspect of our motivation to write the present paper, for as it was mentioned in the introduction and it will become transparent subsequently, one of the main concerns of the present work is with finitary (coarse) and quantal localizations of the topological (*ie*,  $C^0$ ) (Raptis, 2000*a*), as well as the differential (*ie*,  $C^\infty$ -smooth) (Raptis and Zapatrin, 2000), spacetime observables in a curved finitary causal and quantal situation. At least, our central aim is to model ‘coarse (perturbing) quantal acts (operations) of localization (local determination) of the main gravitational observable  $\mathcal{A}$ ’. See sections 5 and 6 for more discussion of this.

<sup>49</sup>For even more technical details and analytical discussion, read the next section.

<sup>50</sup>For more on this demand for ‘local quantum causality’ the reader is referred to (Raptis, 2000*b,c*). We will return to it in the next three sections.

<sup>51</sup>To be represented by the finitary connection  $\mathcal{A}_n$  on the gauged finsheaf of qausets in section 5.

<sup>52</sup>In short, ‘the physical law of classical gravity is mathematically modeled by a differential equation’.

same way that the CEP foreshadows a non-trivial connection (and its associated curvature) in the smooth continuum-the main local dynamical variable of GR as a gauge theory in  $\mathcal{P}$ , so FLP, by ‘cutting-off’ the transitivity property of ‘ $\rightarrow$ ’ furnishes us with the crucial idea of how to model the dynamics of a contiguous (local) quantum causality in a curved finitary causal space, namely, one must define a non-trivial finitary connection (and its associated curvature) on a finsheaf of qausets over it. This connection, in turn, like its smooth spin-Lorentzian counterpart  $\mathcal{A}$  on  $\mathcal{P}$  over  $M$  respects local relativistic causality<sup>53</sup>, should somehow respect the local quantum causal connections in the qauset fibers<sup>54</sup>. This highlights and anticipates two very important aspects of the present paper:

(a) The finitary connection  $\mathcal{A}_n$  (and its associated curvature) derives from the local algebraic structure of the finsheaf of qausets<sup>55</sup>. Thus, our scheme allows for a purely algebraic and local definition of connection (and curvature) without reference to a background geometric base space<sup>56</sup> which will only serve as a surrogate host of  $\mathcal{A}_n$  and which will have to be discarded, or at least be regarded as being physically insignificant, at the quantal level, only to be recovered as a fixed inert (non-dynamical) geometrical structure at the classical limit of an inverse system of curved finsheaves of qausets<sup>57</sup>.

(b) A sheaf (and a non-trivial connection on it) is the ‘right’ (*ie*, the appropriate and natural) mathematical structure for modeling the dynamics (*ie*, the curving) of local quantum causality, since, by definition, a sheaf is a local homeomorphism (Bredon, 1967, Mallios, 1998, Raptis, 2000*a*), so that a  $\mathbf{G}_n$ -finsheaf of qausets by definition respects the reticular local quantum causal topology of the qauset stalks, while a non-flat  $\mathbf{g}_n$ -valued connection  $\mathcal{A}_n$  on it effectively encodes the ‘local twisting’ (curving) of these stalks, thus it represents the dynamics of a locally finite quantum causality. We will return to these issues more analytically in the next three sections.

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<sup>53</sup>Since it preserves the Minkowski lightcone soldered (with origin) at each point-event  $x$  of  $M$ -the Minkowski lightcone in each fiber space  $\mathcal{M}_x$  of  $\mathcal{P}$ .

<sup>54</sup>That is to say, it should respect the generating or ‘germ’ relation  $\vec{\rho}$  of the Rota quantum causal topology of the qauset stalk of the finsheaf in focus. We define  $\vec{\rho}$  in the next section and germs of (sections of) finsheaves of qausets in section 4.

<sup>55</sup>That is to say, from the algebraic structure of the quantal and causal incidence algebras-stalks of the finsheaf in focus.

<sup>56</sup>This is in glaring contrast to the situation in the curved geometrical manifold  $M$  of GR where connection is identified with a parallel transporter (of smooth tensor fields) along smooth finite spacetime curves, while its associated curvature measures the anholonomy of such parallel transports around smooth finite spacetime loops (Göckeler and Schücker, 1991). Certainly, both are non-local geometric conceptions of  $\mathcal{A}$  and its  $\mathcal{F}$ .

<sup>57</sup>See discussion in the introduction and sections 5, 6.



We close this section by giving the analogue of the kinematical Coherent Local Superposition Principle in (Finkelstein, 1988, 1991) for our finsheaves of qausets.

*Finitary Local Superposition Principle (FLSP)*: Stalk-wise in a finsheaf of qausets the latter superpose coherently. It follows from the FLRP that the  $\mathbf{g}_n$ -valued connection  $\mathcal{A}_n$  preserves this ‘stalk-wise quantum coherence of qausets’<sup>58</sup>.

In the next section we present an algebraic approach to flat (*ie*, non-dynamical, non-gauged) local quantum causality, while in sections 4 and 5 we motivate the fin-sheaf theoretic point of view and we study a curved principal finsheaf of qausets, respectively.

### 3. FINITARY SUBSTITUTES AND THEIR FLAT QUANTUM CAUSAL RELATIVES

In this section we motivate the modeling of qausets after incidence algebras (Raptis, 2000*b*), so as to prepare the reader for our representing the stalks of a finsheaf of qausets over some curved finitary causal space as such Rota algebras in section 5. The relevance of qauset theory to the problem of discrete Lorentzian (quantum) gravity is also discussed. In particular, we approach the issue of ‘discrete locality’ or ‘finitary local causality’ via qausets. We quote the main result from (Raptis, 2000*b*) that qausets are sound models of a local and quantal version of the causets of (Bombelli *et al.*, 1987, Sorkin, 1990, 1995) and use it as a theoretical basis to implement the FLP of the previous section, as well as to introduce the central physical idea for curving local quantum causality in section 5 by localizing or gauging a finsheaf of qausets (section 4), thus also realize the FEP of the previous section.

The topological discretization of continuous spacetime (Sorkin, 1991) has as its main aim the substitution of a continuum of events by some finitary, but topologically equivalent, structure. The latter is seen to be a  $T_0$  poset. Such a finitary substitute for the continuous spacetime may be viewed as an approximation of its

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<sup>58</sup>Since  $\mathcal{A}_n$  takes values in the reticular (and quantal) algebra  $\mathbf{g}_n$  of Rota algebra homomorphisms which, in turn, by the functorial equivalence between the category of finitary posets/poset morphisms (or its corresponding category of locally finite causets/causal morphisms) and the category of incidence Rota algebras/Rota homomorphisms (or its corresponding category of qausets/qauset homomorphisms) (Stanley, 1986, Raptis and Zapatrin, 2000, Zapatrin, 2000), it may be regarded as the reticular and quantal version of Zeeman’s (1964) Lie algebra  $\ell^+$  of orthochronous Lorentz transformations (*ie*, the infinitesimal causal automorphisms) of the Minkowski continuum  $\mathcal{M}$  regarded as a poset causal space. We will return to this remark in sections 4-6, but the upshot is that as a linear operator-valued map,  $\mathcal{A}_n$  will preserve the local linear structure stalk-wise, hence, the local quantum coherence or quantum interference of qausets.

continuous counterpart, but one of physical significance, since it seems both theoretically and experimentally lame to assume a continuum as a sound model of what we actually experience (*ie*, record in the laboratory) as ‘spacetime’ (Raptis and Zapatrin, 2000). The theoretical weakness of such an assumption is the continuous infinity of events that one is in principle able to pack into a finite spacetime volume resulting in the unphysical infinities that plague classical and quantum field theory<sup>59</sup>. The experimental weakness of the continuous model of spacetime is that it undermines the operational significance of our actual spacetime experiments, namely, the fact that we record a finite number of events during experimental operations of finite duration in laboratories of finite size; altogether, in experiments of finite spatiotemporal extent (Raptis and Zapatrin, 2000). Also, from a pragmatic point of view, our localizations (*ie*, determinations of the loci) of events are coarse or ‘approximate’ and inflict uncontrollable perturbations to the structure of spacetime<sup>60</sup>, thus our rough, because dynamically perturbing, measurements of events may as well be represented by open sets about them (Sorkin, 1991, 1995, Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000*a*).

Of course, the discrete character of such finitary approximations of a continuous spacetime ties well with the reticular and finite characteristics that a cogent quantal description of spacetime structure ought to have. Thus, if anything, topological discretizations should prove useful in modeling the structure and dynamics of spacetime at quantum scales (Raptis and Zapatrin, 2000). It must be stressed however that such a contribution to our quest for a sound quantum theory of gravity is not mandatory from the point of view of GR—the classical theory of gravity, since in the latter the topology of spacetime is fixed to that of a locally Euclidean manifold<sup>61</sup>, while only the Lorentzian metric on it is assumed to be a dynamically variable entity. Effectively,  $g_{\mu\nu}$  is the sole ‘observable’ in GR. However, it seems rather *ad hoc* and unreasonably limited in view of the persisting and pestilential problem of the quantum localization of spacetime events to assume that only the metric, but not the topological structure of the world, is subject to (quantum) dynamical fluctuations and variations. Such a theory of ‘spacetime foam’, that is to say, of a dynamically fluctuating quantum spacetime topology, has been aired for quite some time now (Wheeler, 1964), and it is akin in spirit to the topological discretizations developed in (Sorkin, 1991), as well as to their quantal relatives in (Raptis and Zapatrin, 2000)<sup>62</sup>.

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<sup>59</sup>See opening quotation by Finkelstein (1991).

<sup>60</sup>Even more so in our scenario where spacetime is assumed to be fundamentally a quantum system.

<sup>61</sup>That is to say, the spacetime of GR is assumed to be locally homeomorphic to the ‘frozen’ (*ie*, non-varying) Euclidean continuum  $\mathbb{R}^4$  (see opening quotation).

<sup>62</sup>See (Sorkin, 1995) for some discussion on this affinity.

On the other hand, in view of the unphysical, non-dynamical, non-relativistic, space-like nature of the constant, two-way, spatial connections between events that define the (locally) Euclidean topology of the classical spacetime continuum  $M$ , there is an important affinity between our quest for a dynamical theory of (local) quantum causal topology and the problem of constructing a reasonable quantum theory of gravity. To understand this close relationship, we must change focus of enquiry from a theory of spatial Euclidean connections between points<sup>63</sup> to a more physical, because relativistic, temporal or causal spacetime topology<sup>64</sup> between events as the quotation opening this paper and the following one from (Finkelstein, 1988) suggest:

It is therefore crucial to make use of the proper physical topology. The usual combinatory topology is founded on a symmetric concept of connection derived from experience with Riemannian and ultimately Euclidean geometry. It assumes the existence of spatial connections and puts them on the same footing as timelike ones, when (in the absence of any signs of tachyons) it is quite doubtful that either exist at all. A relativistic topology should deal with causal connections among events, not spatial connections among objects. At the continuum level, the Alexandroff<sup>65</sup> topology is already suitably relativistic. It is thus only necessary to construct a relativistic discrete topology and homology theory on the basis of the causal connection  $\mathbf{c}$ <sup>66</sup>.

As it was mentioned in the previous section, in GR, the gravitational potential, which is identified with the metric  $g_{\mu\nu}$  of spacetime, may also be thought of as encoding complete information about the local causal relations between events<sup>67</sup>. Thus, GR may also be interpreted as the dynamical theory of ‘the field of local causality’. It follows that a quantum theorems of the dynamics of causal connections between spacetime events may lead to, if not just give us invaluable clues about, a ‘classically conceived quantum theory of gravity’-‘the quantization of the gravitational field

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<sup>63</sup>Ultimately, a ‘topo-’ or ‘choro-logy’ (Greek for ‘a theory of space’).

<sup>64</sup>Ultimately, a ‘chrono-logy’ (Greek for ‘a theory of time’).

<sup>65</sup>This is the A. D. Alexandrov (1956, 1967) (our footnote).

<sup>66</sup>‘ $\mathbf{c}$ ’ in (Finkelstein, 1988) stands for an atomistic, local, dynamical relation that defines a dynamically variable causal topology between events. Subsequent ‘algebraization’ of  $\mathbf{c}$  results in a model for the dynamics of local quantum causality, called ‘the quantum causal net’, which is quite similar to the one that we propose here. See also (Raptis, 2000b).

<sup>67</sup>That is to say, at every event  $x$ ,  $g_{\mu\nu}(x)$ , which can be reduced to the Minkowski metric  $\eta_{\mu\nu}$  by the CEP, delimits the lightcone at  $x$ , which in turn determines the possible (kinematical) causal relations between it and its local neighbors in the Minkowski space tangent to it (‘infinitesimal locality’ or ‘local causality’). Albeit, as Finkelstein successfully observed in (1988), Einstein, following Riemann, ‘metrized’ causality in GR instead of ‘topologizing’ it.

$g_{\mu\nu}(x)$  of GR<sup>68</sup>. In short, there probably is a way from a dynamical theory of local quantum causality to the graviton, but not the other way around<sup>69</sup>. A full-fledged non-commutative topology for curved (*ie*, dynamical) local quantum causality will be rigorously formulated in the scheme theoretic language of modern algebraic geometry and its categorical outgrowth, topos theory, in a coming paper (Raptis, 2000*e*).

However, it must be stressed that it is quite clear, at least from a ‘gedanken-experimental’ point of view, why GR and Quantum Theory are incompatible: the more accurately one may try to determine the spacetime metric<sup>70</sup>, the more energy one must employ, the stronger the dynamical perturbations inflicted on it, the higher the uncertainty of its local determination<sup>71</sup>. Another way to say this is that we can not distinguish spacetime events<sup>72</sup> at a resolution higher than the Planck length ( $l_P \approx 10^{-33}cm$ ) without creating a black hole which, in return, ‘fuzzies’ their separation in some sense<sup>73</sup>. This limitation alone is sufficient to motivate some kind of ‘topological foam’ conception of spacetime at quantum scales (Wheeler, 1964). An analogous incompatibility (of physical principles) that may hinder the development of a quantum theory of the dynamics of a finitary causality has not been predicted yet. We hope that such a fundamental conflict of physical principles will be absent *ab initio* from an innately locally finite dynamical theory of quantum causality<sup>74</sup>, or at least from the relevant kinematics for such a theory, like the one that we will propose in section 5.

This lengthy prolegomenon to the introduction of the essentially flat qausets in (Raptis, 2000*b*) highlights two important aspects of our present endeavor: (a) our locally finite, and subsequently to be gauged, qausets, may evade *ab initio* the infinities of Quantum GR on a smooth manifold, and (b) as quantum causal-topological

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<sup>68</sup>Otherwise known as ‘Quantum General Relativity’. Again, it is Finkelstein who successfully observed that gravity is of an essentially local causal-topological nature (Finkelstein, 1988).

<sup>69</sup>Read the end of the quotation from Bombelli *et al.* (1987) given in the introduction and (Sorkin, 1990, 1995).

<sup>70</sup>That is to say, ‘localize’ it.

<sup>71</sup>That is to say, the CEP on which GR is based comes straight into conflict with the Uncertainty Principle on which Quantum Theory is founded (Candelas and Sciama, 1983, Donoghue *et al.*, 1984, 1985).

<sup>72</sup>Or equivalently, measure the distance of their separation via the gravitational potential  $g_{\mu\nu}$ .

<sup>73</sup>This is the quintessential paradox of event-localization that makes the conception of a quantum theory of gravity hard even in principle: the more accurately we try to localize spacetime events, the more we blur them, so that our sharpest determinations of them can be modeled after coarse, rough, fuzzy, ‘dynamically fluctuating’ open neighborhoods about them as in (Sorkin, 1991, 1995, Zapatrin, 1998, Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000*a*).

<sup>74</sup>In other words, a ‘dynamical finitary quantum causal topology’.

structures, they grapple with the problem of the structure and dynamics of spacetime at quantum scales at a level deeper than Quantum GR-proper which is supposed to study the quantum aspects of the dynamics of the metrical structure of the world, because we have seen already at the classical level that causality, as a partial order, and its morphisms, determine the geometric structure of flat Minkowski space and its symmetries (Robb, 1914, Alexandrov, 1956, 1967, Zeeman, 1964, 1967, Bombelli and Meyer, 1989, Sorkin, 1990). Afterall, as Bombelli *et al.* (1987) successfully observed in the excerpt presented in the introduction, it is such a model for events and their causal relations that uniquely determines spacetime as a 4-dimensional, continuous ( $C^0$ ), differential ( $C^\infty$ -smooth) and Lorentzian metric (*ie*, of signature  $\pm 2$ ) manifold.

We commence our brief review of gausets by first presenting elements of finitary substitutes for continuous spacetime topology (Sorkin, 1991, Raptis and Zapatrin, 2000, Raptis, 2000*a,b*). Let  $X$  be a bounded region in a continuous spacetime manifold  $M^{75}$  and  $\mathcal{U} = \{U\}$  a locally finite open cover of it<sup>76</sup>. The boundedness of  $X$  stands for the pragmatic restriction of our experimental discourse with spacetime in laboratories of finite spatio-temporal extent; while, its locally finite open covering by  $\mathcal{U}$  reflects the fact that within this experimental activity of finite duration and spatial extension ‘for all practical purposes’ we record a finite number of events coarsely or ‘roughly’, that is to say, by determining a finite number of open sets about them (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000*a*).

Any two points  $x$  and  $y$  of  $X$  are indistinguishable with respect to its locally finite open cover  $\mathcal{U}$  if  $\forall U \in \mathcal{U} : x \in U \Leftrightarrow y \in U$ . Indistinguishability with respect to the subtopology  $\mathcal{T}(\mathcal{U})^{77}$  of  $X$  is an equivalence relation on the latter’s points and is symbolized by  $\sim^{\mathcal{U}}$ . Taking the quotient  $X / \sim^{\mathcal{U}} =: F$  results in the substitution of  $X$  by a space  $F$  consisting of equivalence classes of its points, whereby two points in the same equivalence class are covered by (*ie*, belong to) the same, finite in number, open neighborhoods  $U$  of  $\mathcal{U}$ , thus are indistinguishable by (our coarse observations in) it.

Let  $x$  and  $y$  be points belonging to two distinct equivalence classes in  $F$ . Consider the smallest open sets in the subtopology  $\mathcal{T}(\mathcal{U})$  of  $X$  containing  $x$  and  $y$  respectively given by:  $\Lambda(x) := \cap\{U \in \mathcal{U} : x \in U\}$  and  $\Lambda(y) := \cap\{U \in \mathcal{U} : y \in U\}$ . Define the relation  $\rightarrow$  between  $x$  and  $y$  as follows:  $x \rightarrow y \Leftrightarrow \Lambda(x) \subset \Lambda(y) \Leftrightarrow x \in \Lambda(y)$ . Then assume that  $x \sim^{\mathcal{U}} y$  in the previous paragraph stands for  $x \rightarrow y$  and  $y \rightarrow x$ <sup>78</sup>.  $\rightarrow$  is a

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<sup>75</sup>By ‘bounded’ we mean ‘relatively compact’ (*ie*, a region whose closure is compact). By ‘continuous’ we mean the  $C^0$  aspects of classical spacetime (*ie*, spacetime as a topological manifold).

<sup>76</sup>That is to say, every point-event  $x$  in  $X$  has an open neighborhood  $O(x)$  that meets only a finite number of open sets  $U$  in  $\mathcal{U}$ .

<sup>77</sup> $\mathcal{T}$  consists of arbitrary unions of finite intersections of the open sets in  $\mathcal{U}$ .

<sup>78</sup>That is to say,  $x$  and  $y$  have the same smallest open neighborhood about them with respect to

partial order on  $F$  and the continuous  $X$  has been effectively substituted by the finitary  $F$  which is a  $T_0$  topological space having the structure of a poset (Sorkin, 1991). Sorkin uses the finitary topological and partial order theoretic languages interchangeably exactly due to this equivalence between  $T_0$  finitary substitutes and posets. For future purposes we distill this to the following statement: in (Sorkin, 1991) a partial order is interpreted topologically. We call it ‘topological partial order’ and the poset encoding it ‘topological poset’ (Raptis, 2000b).

Topological posets have an equivalent representation as simplicial complexes (Raptis and Zapatrin, 2000). One may represent a finitary spacetime substitute by a simplicial complex by considering the so-called nerves of locally finite open covers of  $X$  (Alexandrov, 1956, 1961)<sup>79</sup>. We may recall that the nerve  $\mathcal{N}$  of a covering  $\mathcal{U}$  of a manifold  $X$  is the simplicial complex whose vertices are the elements of  $\mathcal{U}$  and whose simplices are formed according to the following rule. A set of vertices, that is to say, elements of the locally finite covering  $\{U_0, \dots, U_k\}$  form a  $k$ -simplex of  $\mathcal{N}$  if and only if they have nonempty intersection:

$$\{U_0, \dots, U_k\} \in \mathcal{N} \Leftrightarrow U_0 \cap U_1 \cap \dots \cap U_k \neq \emptyset$$

Any nerve  $\mathcal{N}$ , being a simplex, can be equivalently treated as a poset, denoted also by  $\mathcal{N}$ . The points of the poset  $\mathcal{N}$  are the simplices of the complex  $\mathcal{N}$ , and the arrows are drawn according to the rule:

$$p \rightarrow q \Leftrightarrow p \text{ is a face of } q.$$

In the nondegenerate cases, the posets associated with Alexandrov nerves and those produced by Sorkin’s ‘equivalence algorithm’ yielding  $F$  from  $X$  relative to  $\mathcal{U}$  as described above are the same, so that both are ‘topological posets’ according to our denomination of  $F$ .

In (Raptis and Zapatrin, 2000) an algebraic representation of topological posets was presented using the so-called Rota incidence algebras associated with posets (Rota, 1968). The Rota incidence algebra  $\Omega$  of a poset  $P$  was defined there by using Dirac’s quantal ket-bra notation as follows

$$\Omega(P) = \text{span}\{|p\rangle\langle q| : p \rightarrow q \in P\},$$

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$\mathcal{T}(\mathcal{U})$ .

<sup>79</sup>This is the P. S. Alexandrov.

with product between two of its ket-bras given by

$$|p\rangle\langle q| \cdot |r\rangle\langle s| = |p\rangle\langle q|r\rangle\langle s| = \langle q|r\rangle \cdot |p\rangle\langle s| = \begin{cases} |p\rangle\langle s| & , \text{ if } q = r \\ 0 & \text{ otherwise.} \end{cases}$$

Evidently, for the definition of the product in  $\Omega$ , the transitivity of the partial order  $\rightarrow$  in  $P$  is used.  $\Omega(P)$ , defined thus, is straightforwardly verified to be an associative algebra<sup>80</sup>. When  $P$  is a finitary topological poset in the sense of Sorkin (1991), its associated incidence algebra is called ‘topological incidence algebra’ (Raptis, 2000b).

We may define purely algebraically a topology on any incidence algebra  $\Omega$  associated with a poset  $P$  by considering its primitive spectrum  $\mathcal{S}$  consisting of the kernels of its irreducible representations, which are primitive ideals in it, in the following way according to Breslav *et al.* (1999): with every point  $p$  in  $P$  the following ideal in  $\Omega$  is defined

$$I_p = \text{span}\{|q\rangle\langle r| : |q\rangle\langle r| \neq |p\rangle\langle p|\},$$

so that the Rota topology of  $\Omega(P)$  is generated by the following relation  $\rho$  between ‘points’  $I_p$  and  $I_q$  in its primitive spectrum  $\mathcal{S}$

$$I_p \rho I_q \Leftrightarrow I_p I_q (\neq I_q I_p) \stackrel{\neq}{\subset} I_p \cap I_q.$$

It has been shown that the Sorkin topology of a topological poset  $P$  is the same as the Rota topology of its associated topological incidence algebra  $\Omega(P)$  exactly when the generating relation  $\rho$  for the latter is the transitive reduction  $\xrightarrow{*}$  of the partial order arrows  $\rightarrow$  in  $P$  (Breslav *et al.*, 1999, Raptis, 2000b)<sup>81</sup>. This means essentially

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<sup>80</sup>The associativity of the product of the incidence algebra  $\Omega$  is due to the transitivity of the partial order  $\rightarrow$  of its associated poset  $P$ . As we saw in section 2, it is precisely the latter property of causality, when modelled after  $\rightarrow$ , that is responsible for the (global) flatness of Minkowski space determined by  $\rightarrow$ . It follows that a localization or gauging of causets and their corresponding qausets in order to curve them, by providing a connection on a principal finsheaf of theirs, will ‘cut-off’ the transitivity of the causets and the associativity of their corresponding qausets, and will restrict it locally (*ie*, stalk-wise) in the finsheaf (section 5) thus implement the FEP of section 2. We will briefly return to this conception of global non-associativity in a curved topos of finsheaves of qausets in ( $f'$ ) of section 6.

<sup>81</sup>That is to say,  $I_p$  is  $\rho$ -related to  $I_q$  if and only if  $(p \xrightarrow{*} q) \Leftrightarrow [(p \rightarrow q) \wedge (\nexists r : p \rightarrow r \rightarrow q)]$ ;  $p, q, r \in P$  (*ie*, only for immediately connected contiguous vertices in  $P$ ).

that the ‘germ-relations’ for the Rota topology on the algebra  $\Omega$  associated with the finitary topology  $P$  are precisely the immediate arrows  $\xrightarrow{*}$  in the latter topological poset. This is an important observation to be used shortly in order to define in a similar way the germs of quantum causal relations in a qauset with respect to which finsheaves of the latter will be defined in the next section as structures that preserve these local quantum causal topological ‘germ relations’<sup>82</sup>.

To this end, we give the definition of qausets. A causet is defined in (Bombelli *et al.*, 1987) as “*a locally finite set of points endowed with a partial order corresponding to the macroscopic relation that defines past and future*”. Local finiteness may be defined as follows: use  $\rightarrow$  of a poset  $P$ , interpreted now as a causal relation on the causet, to redefine  $\Lambda(x)$  for some  $x \in P$  as  $\Lambda(x) = \{y \in P : y \rightarrow x\}$ , and dually  $V(x) = \{y \in P : x \rightarrow y\}$ .  $\Lambda(x)$  is the ‘causal past’ of the event  $x$ , while  $V(x)$  its ‘causal future’. Then, local finiteness requires the so-called Alexandrov set  $V(x) \cap \Lambda(y)$  to be finite for all  $x, y \in P$  such that  $x \in \Lambda(y)$ . In other words, only a finite number of events ‘causally mediate’ between any two events  $x$  and  $y$ , with  $x \rightarrow y$ , of the causet  $P$ . In a sense, the finitariness of the topological posets translates by Sorkin’s semantic switch to the local finiteness of causal sets, although it must be stressed that the physical theories that they support, the discretization of topological manifolds in (Sorkin, 1991) and causet theory in (Bombelli *et al.*, 1987 and Sorkin, 1995) respectively, are quite different in motivation, scope and aim (Sorkin, 1990, 1991, 1995, Raptis, 2000b)<sup>83</sup>.

On the other hand, it was Sorkin who first insisted on a change of physical interpretation for the partial order  $\rightarrow$  of finitary posets  $P$  “*from a relation encoding topological information about bounded regions of continuous spacetimes, to one that stands for the relation of causal succession between spacetime events*” (Sorkin, 1995). The following quotation from that paper is very telling indeed<sup>84</sup>:

Still, the order inhering in the finite topological space seemed to be very different from the so-called causal order defining past and future. It had only a topological meaning but not (directly anyway) a causal one. In fact the big problem with the finite topological space was that it seemed to lack the information which would allow

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<sup>82</sup>That is to say, as ‘local quantum causal homeomorphisms’. In (Raptis, 2000e) this ‘topologization’ of the primitive spectra of incidence Rota algebras associated with Bombelli *et al.*’s causets, that is, the qausets in (Raptis, 2000b), will be the chief motivation for the definition of primitive finitary spacetime schemes (finschemes) of qausets and to study their localization properties, their non-commutative topological aspects, as well as the organization of these finschemes into a topos-like structure called ‘quantum topos’ (see also  $f'$  in section 6).

<sup>83</sup>See also discussion in ( $d'$ ) of section 6.

<sup>84</sup>This quotation can be also found in (Raptis, 2000b).



it to give rise to the continuum in all its aspects, not just in the topological aspect, but with its metrical (and therefore its causal) properties as well...The way out of the impasse involved a conceptual jump in which the formal mathematical structure remained constant, but its physical interpretation changed from a topological to a causal one...The essential realization then was that, although order interpreted as topology seemed to lack the metric information needed to describe gravity, the very same order reinterpreted as a causal relationship, did possess information in a quite straightforward sense...In fact it took me several years to give up the idea of order-as-topology and adopt the causal set alternative as the one I had been searching for...

In (Raptis, 2000*b*), this ‘semantic switch’ was evoked to reinterpret the incidence algebras associated with the finitary posets in (Raptis and Zapatin, 2000) from topological to causal. Thus, causal incidence algebras were defined as the  $\Omega$ s associated with finitary posets  $P$  when the latter are interpreted as causets *à la* Bombelli *et al.* (1987). Of course, in our pursuit of a cogent quantum theory of the dynamics of causality and, *in extenso*, of gravity, such a change of physical meaning of finitary partial orders from spatial/choro-logical/topo-logical to temporal/chrono-logical/causal is very welcome for the reasons given earlier in this section.

Finally, in (Raptis, 2000*b*) the quantum physical interpretation given to topological incidence algebras in (Raptis and Zapatin, 2000) was also given directly to causal incidence algebras. Thus, qausets were defined as the causally and quantally interpreted incidence algebras associated with poset finitary substitutes of continuous spacetimes<sup>85</sup>. It follows that the generator  $\xrightarrow{*}$  of topological relations in the topological posets of Sorkin becomes the germ  $\vec{\rho}$  of quantum causal relations in qausets<sup>86</sup>. Its interpretation is as ‘immediate quantum causality’<sup>87</sup> and it is exactly due to its natural Rota-algebraic representation that qausets are (operationally) sound models

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<sup>85</sup>Thus, unlike Regge (1961) who metrized the simplicial skeletons of spacetime and arrived at a simplicial gravity in the same spirit as to how Einstein identified the gravitational potential with the spacetime metric in the continuum theory, we, following Sorkin (1995), causalize the simplicial-poset topological discretizations of spacetime in (Alexandrov, 1956, 1961, Sorkin, 1991) and their quantum relatives in (Raptis and Zapatin, 2000) with an aim to arrive at a finitary quantum causal version of gravity. Einstein and Regge metrized topology, we causalize it (*ie*, we re-interpret topology causally), and subsequently (section 5) we regard it as a dynamical local variable as the opening quotation from Finkelstein (1991) motivated us to).

<sup>86</sup>Due to its causal instead of topological meaning, we are going to write  $\vec{\rho}$  instead of  $\rho$  from now on for the local quantum causal topological variable.

<sup>87</sup>The epithet ‘quantum’ refers precisely to the possibility for coherent quantum superpositions of the causal arrows of  $P$  in its associated incidence algebra  $\Omega(P)$  (Raptis and Zapatin, 2000, Raptis, 2000*b*).

of quantum causal spaces (Finkelstein, 1988, Raptis, 2000b).  $\vec{\rho}$  is the algebraic correspondent in the causal incidence algebra  $\vec{\Omega}$  of the immediate causal relation  $\xrightarrow{*}$  of its associated causet  $\vec{P}$ <sup>88</sup>.

The immediate quantum causality represented by  $\vec{\rho}$  in the incidence algebra associated with a causet is ideal for implementing the FLP of the previous section. In particular, it explicitly shows that the physically significant, because local, quantum causality is the relation  $\xrightarrow{*}$  between immediately separated events in a finitary space-time  $X$  (Finkelstein, 1988, Raptis, 2000b). Its non-local transitive closure, the partial order  $\rightarrow$  in the associated causet  $\vec{P}$ , generates  $\vec{P}$ 's (globally) inertial Minkowskian causal topology which, being a finitary poset, essentially determines a locally finite version of flat Minkowski space and its global orthochronous Lorentz symmetries (Robb, 1914, Alexandrov, 1956, 1967, Zeeman, 1964, 1967).

It follows that in order to curve qausets, a gauged or localized version of  $\vec{\rho}$  must be employed, that is to say, we should consider a dynamical local quantum causal connection relation that only locally (*ie*, event-wise)<sup>89</sup> reduces to a transitive partial order—the inertial Minkowskian causality of a reticular and quantal Minkowski space (as a qauset) according to the FEP. In turn, this means that only the transitive reduction  $\xrightarrow{*}$  of the flat inertial causality  $\rightarrow$  will be ‘operative’ (*ie*, physically significant) in a curved finitary quantum causal space. We will model this conjecture by a non-flat connection  $\mathcal{A}_n$  on a finsheaf of qausets in section 5. In a sense, such a non-trivial connection  $\mathcal{A}_n$  will be seen to ‘cut-off’ the transitivity of causality as a partial order  $\rightarrow$ , so as to restrict the latter to immediate causal neighborhoods of events defined by  $\xrightarrow{*}$ , thus respect the FLP<sup>90</sup>.

We conclude the present section by discussing briefly two relatively important aspects of qausets, one physical, the other mathematical.

The physical aspect of qausets pertains to their operational significance<sup>91</sup>. While

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<sup>88</sup>Notice again the arrow over  $P$  when the latter is interpreted as a causet rather than as a topological poset.

<sup>89</sup>As we said in section 2 when we discussed the FLP, in the finsheaf theoretic context of section 5 locality will correspond to ‘stalk-wise over an event in a curved finitary causal space’.

<sup>90</sup>Hence,  $\mathcal{A}_n$  will also qualify as a finitary (and quantal) local dynamical variable (*ie*, ‘observable’) in the aforementioned gauged finsheaf of qausets.  $\mathcal{A}_n$ 's ‘purely algebraic character’ mentioned before (*ie*, its essentially non-commitment to any geometric background base spacetime whatsoever) is that it represents the dynamics of the local quantum causal topology  $\vec{\rho}$  of the qauset stalks of the finsheaf and the latter is purely algebraically defined in the spectra of these stalks (see section 5). The purely algebraico-categorical nature of  $\mathcal{A}_n$  will be also pronounced in our scheme theoretic presentation of qausets in (Raptis, 2000e).

<sup>91</sup>Here, we refer to the classical definition of operationality by Bridgman (1936) according to which: “*in dealing with physical situations, the operations that give meaning to our physical concepts should*

the operational soundness of quantum discretized spacetimes has been fairly established (Raptis and Zapatin, 2000) in that we have a sound operational interpretation of quantal topological incidence algebras<sup>92</sup>, we still lack such an account for gausets.

Now, GR's operational significance can be summarized in the following:  $g_{\mu\nu}(x)$ , which mathematically represents the local gravitational potential, is supposed to encode all the information about our local experimental tampering with spacetime events via synchronized clocks and equicalibrated rulers, so that, in principle, from the data of such a local experimental activity, one can construct the metric tensor at a neighborhood of an event (the latter serving as the origin of our laboratory/frame). In such an operational account, there is little room left for a 'passive' realistic interpretation of the gravitational field as an independent entity or substance 'out there' whose interaction with our instruments yields readings of events. The operational approach is in an important sense more active in that it entails that spacetime attributes are extracted from 'it'<sup>93</sup> by our very experimental actions on (*ie*, our planned, coordinated and controlled observations of) 'it'. Also, this seems to be more in accord with

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*properly be physical operations, actually carried out*".

<sup>92</sup>Briefly, the operational meaning of quantal incidence algebras goes as follows: the finitary substitutes of continuous spacetime, be it posets as in Sorkin (1991) or simplicial complexes as in Alexandrov (1956, 1961), stand for coarse (*ie*, approximate) determinations of the loci of spacetime events. Since these measurements are imperfect and, in a quantum sense, they inflict uncontrollable perturbations on spacetime, what they actually determine is 'rough', 'blurry' or 'fuzzy' regions about events that can be modeled after open sets. The latter are supposed to model the irreducible quantum uncertainties of space-time localizations that are expected at scales smaller than Planck's  $l_P \approx 10^{-33}cm$  and  $t_P \approx 10^{-42}s$  (Capozziello *et al.*, 2000). By covering the region of spacetime under experimental focus by a finite number of such neighborhoods, and by keeping track of the mutual intersections of these coarse determinations (*ie*, their nerves), enables one to build the Alexandrov simplicial complexes of such an experimental activity, or their topologically equivalent finitary posets of Sorkin (Raptis and Zapatin, 2000). This is a formal account of "*what we actually do to produce spacetime by our measurements*" (Sorkin, 1995)-the essence of operationality. In the context of quantum topological incidence algebras, sound operational meaning has been given to the corresponding algebraic structures and it all is operationally sound (Raptis and Zapatin, 2000). In the latter reference, Heisenberg's algebraic approach to quantum theory, that points to a more general and abstract conception of operationality where quantum physical operations are organized into an algebra (call it 'quantum operationality' to distinguish it from Bridgman's classical conception of this notion that is not based on a specific mathematical modeling of the actual physical operations), and according to which "the physical operations that give meaning to our conceptions about the quantum realm, ultimately, our own experimental actions on the quantum system that define (*ie*, prepare and register (Finkelstein, 1996)) its observable quantities, should be modeled algebraically", was used to interpret quantum physically the incidence algebras associated with Sorkin's (1991) finitary topological posets.

<sup>93</sup>'It' referring to the physical system 'spacetime'.

the observer-dependent conception of physical reality that Quantum Theory supports (Finkelstein, 1996)<sup>94</sup>.

For the causets of Bombelli *et al.* (1987) and Sorkin (1990), Sorkin (1995) contended that an operational interpretation is rather unnatural and lame. On the other hand, in view of the algebraic structure of qausets and the sound quantum-operational interpretation *à la* Raptis and Zapatrin (2000) that their topological counterparts were given, and because as we mentioned in section 2 the local field of gravity  $g_{\mu\nu}(x)$  can also be interpreted as the dynamical field of the local causal topology of spacetime, we still hope for a sound operational interpretation of them. At the end of the next section we present our first attempt at a sound operational interpretation of the locally finite quantum causality encoded in qausets based on the analogous operational meaning of the finitary poset substitutes of continuous spacetimes and their incidence algebras (Sorkin, 1991, 1995, Raptis and Zapatrin, 2000). A more thorough presentation of the operational character of qausets will be given in a coming paper (Raptis, 2000*d*).

The mathematical aspect of qausets that we would like to discuss next is their differential structure. Recently, there has been vigorous research activity on studying differential calculi on finite sets and the dynamics of networks (Dimakis *et al.*, 1995), as well as on defining some kind of discrete Riemannian geometry on them (Dimakis and Müller-Hoissen, 1999). The main result of such investigations is that with every directed graph a discrete differential calculus may be associated. It follows that for the locally finite posets underlying qausets  $\vec{\Omega}$  (*ie*, the causets  $\vec{P}$  associated with them), which are also (finitary) digraphs, there is a discrete differential calculus associated with them (Raptis and Zapatrin, 2000, Zapatrin, 2000). In this sense, but from a discrete perspective, a partial order determines not only the topological ( $C^0$ ), but also the differential ( $C^\infty$ ) structure of the spacetime manifold with respect to which the Lorentzian  $g_{\mu\nu}$ , which is also determined by causality as a partial order<sup>95</sup>, is then defined as a smooth field<sup>96</sup>.

However, as we noted in the introduction, the Kähler-Cartan type of discrete

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<sup>94</sup>The reader is referred to that book for a deep operational, in fact, pragmatic, ‘unification’ of the basic principles of relativity and quantum theory. Finkelstein elevates the aforementioned ‘quantum operationalism’ of Heisenberg, to a philosophically sound ‘quantum relativistic pragmatism’ which, in a nutshell, holds that our rationalization about the quantum and relativistic realm of Nature and the properties of relativistic quantum systems—our quantum relativistic calculus (logic or set theory) so to speak, is defined by our very experimental (inter)actions on (with), ultimately, by our dynamical transformations (in the general operational sense of actions of preparation, propagation and registration) of, relativistic quanta.

<sup>95</sup>At least locally in a curved spacetime (see section 2).

<sup>96</sup>See the quotation from (Bombelli *et al.*, 1987) in the introduction.

differential operator  $D$  defined in such calculi on finite sets is a flat sort of connection (Mallios, 1998). This is not surprising, since the underlying finite space(time)  $X$  is taken to be a structureless point-set<sup>97</sup>. All the digraphs supporting such calculi are assumed to be transitive, so that if some causal interpretation was given to their arrows, by our heuristic principles of section 2 concerning the relation between an inertial transitive causality and flatness, their corresponding differential calculi should be flat as well<sup>98</sup>. This is the ‘curvature problem’ alluded to in the introduction. To evade it, in section 5 we straightforwardly gauge (a finsheaf of) qausets so that a non-flat connection  $\mathcal{A}_n$  is naturally defined on them. The physical interpretation of such a gauging of the  $D$  of flat qausets to the  $D = D + \mathcal{A}$  of the curved finsheaf of qausets, will be the first essential step towards a finitary, causal and quantal version of Lorentzian gravity.

In the next section we recall the finsheaves from (Raptis, 2000a). Our principal aim is to review the sense in which a finsheaf of continuous maps over Sorkin’s topological posets approximates the sheaf of  $C^0$ -topological observables over a continuous spacetime manifold, then try to ‘read’ a similar physical meaning for a finsheaf of qausets, namely, that they model finitary and quantal replacements of the causal relations between events in a bounded region of flat Minkowski space  $\mathcal{M}$ , as well as the causal nexus of  $C^\infty$ -smooth fields in such a region of the smooth differential manifold  $\mathcal{M}$ . At the same time, the finsheaves of their continuous symmetries may be thought of as locally finite substitutes of the continuous orthochronous Lorentz topological ( $C^0$ ) Lie group manifold  $SO(1,3)^\dagger$ . In such a scenario, not only the operational significance of our own coarse ‘approximations’ of spacetime structure and its dynamics will be highlighted, but also the operational meaning of our rough and dynamically perturbing determinations of its symmetries.

#### 4. FINITARY SPACETIME SHEAVES AND THEIR FLAT QUANTUM CAUSAL DESCENDANTS

In (Raptis, 2000a), a finsheaf  $S_n$  of continuous functions on a bounded region  $X$  of a topological spacetime manifold  $M$  was defined as the sheaf of sections of continuous maps on  $X$  relative to its covering by a locally finite collection of open subsets of  $M$ . Since, as we saw in the previous section, for every such finitary open cover  $\mathcal{U}_n$  of  $X$  a finitary topological poset  $F_n$  was defined and seen to effectively substitute

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<sup>97</sup>In a sense, a kind of disconnected, non-interacting ‘dust’.

<sup>98</sup>That is to say, the differential operators defining such calculi are flat connections in the sense of Mallios (1998).

$X$ , the aforementioned sheaf can be thought of as having  $F_n$  as base space. Thus, we write  $S_n(F_n)$  for such a finsheaf (Raptis, 2000a). Indeed,  $S_n$  was seen to have locally the same finite poset-topology as its base space  $F_n$ <sup>99</sup> (Raptis, 2000a), hence its qualification as a sheaf (Bredon, 1967, Mallios, 1998).

Now, as we briefly alluded to in the introduction, the essential result from (Raptis, 2000a), and the one that qualifies finsheaves as sound approximations of the continuous spacetime observables on  $X$ , is that an inverse system of finsheaves has an inverse limit topological space that is homeomorphic to  $S(X)$ -the sheaf of continuous functions on  $X$ , in the same way that in (Sorkin, 1991) an inverse system of finitary poset substitutes of  $X$  was seen to ‘converge’ to a space that is homeomorphic to the continuous topological manifold  $X$  itself.

To define finsheaves of gausets we adopt from (Raptis and Zapatrin, 2000) the association with every poset finitary substitute  $F_n$  of a bounded spacetime region  $X$ , of a Rota incidence algebra  $\Omega(F_n)$ , as it was shown in the previous section. As  $F_n$  is a topological poset, its associated  $\Omega_n$  is a topological incidence algebra (Raptis, 2000b). As we noted in the previous section, to get the gauset  $\vec{\Omega}_n$  from  $\Omega_n$ , we ‘causalize’ and ‘quantize’ it *à la* Raptis (2000b). As a result of such a causalization, we write  $\vec{\rho}$  for the generating relation of  $\vec{\Omega}_n$ ’s (quantum) causal topology in the same way that  $\rho$  in the previous section was seen to be the generator of  $\Omega$ ’s spatial Rota topology. The significance of  $\vec{\rho}$  is (quantum) causal, while of  $\rho$ , only topological.

Finsheaves of gausets are then defined to be objects  $\vec{S}_n := \vec{\Omega}_n(\vec{F}_n)$ <sup>100</sup>, whereby the local homeomorphism between the base causal set  $\vec{F}_n$  and  $\vec{\Omega}_n$  is now given, in complete analogy to the finsheaf  $S_n(F_n)$  of topological posets in (Raptis, 2000a), as  $p \xrightarrow{*} q \Leftrightarrow I_p \vec{\rho} I_q$ , ( $p, q \in \vec{F}_n$ ,  $I_p, I_q \in \vec{S}(\vec{\Omega}_n)$ )<sup>101</sup>. As it was mentioned in the previous section, the Sorkin poset topology on the topological  $F_n$ , obtained as the transitive closure of the immediate contiguity relation  $\xrightarrow{*}$  between its vertices, is the same as the Rota topology of its associated topological incidence algebra  $\Omega_n$  generated by  $\rho$ <sup>102</sup>, only now, these relations have a directly causal/temporal rather than a topological/spatial significance (Sorkin, 1995, Raptis, 2000b).

Also, in the same way that  $S_n(F_n)$  was seen to be the finsheaf of continuous maps

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<sup>99</sup>Technically speaking,  $S_n$  is locally homeomorphic to the finite topological space  $F_n$  of (Sorkin, 1991).

<sup>100</sup>Note again that we have put an arrow over both  $F_n$  and  $S_n$ , since the partial order relation  $\rightarrow$  is interpreted causally rather than topologically (Sorkin, 1995, Raptis, 2000b,c).

<sup>101</sup>See previous section for relevant definitions and note that the primitive spectrum  $\mathcal{S}$  of the gauset  $\vec{\Omega}_n$  also carries an arrow over it to remind one of its causal meaning.

<sup>102</sup>In other words,  $\rho$  for the Rota topology of  $\Omega_n$  is the transitive reduction of Sorkin’s partial order topological relation  $\rightarrow$  in  $F_n$ .

on the  $F_n$  obtained from  $X$  with respect to its locally finite open cover  $\mathcal{U}_n$  and generated by its (germs of) continuous sections (Raptis, 2000a), we may similarly consider  $\mathbf{G}_n := \mathcal{L}_n(\vec{\Omega}_n)$  to be the finsheaf of local (quantum) causal (auto)morphisms of  $\vec{\Omega}_n$ . We may call  $\mathbf{G}_n$  ‘the finitary spacetime transformation sheaf adjoint to  $\vec{S}_n$ ’<sup>103</sup>. The (germs of) continuous sections of this sheaf are precisely the maps that preserve the local (quantum) causal topology  $\vec{\rho}$  of  $\vec{\Omega}_n$  and by the definition of the latter, they are the  $\vec{\Omega}_n$ -homomorphisms ‘restricted’ to the primitive ideals  $I_p$  and  $I_q$  in them-the Gel’fand ‘point-events’ of the qauset  $\vec{\Omega}_n$ <sup>104</sup> which is the finitary base space of the finsheaf  $\mathbf{G}_n$ . The finsheaf  $\mathbf{G}_n$  consists of the local causal homeomorphisms  $\vec{\lambda}_n$  of the local (quantum) causal topology (generated by)  $\vec{\rho}$  of the qauset  $\vec{\Omega}_n$  which, by the discussion in section 2, constitutes the finitary version of the orthochronous Lorentz group  $L^{+105}$ . Thus, the finsheaf  $\vec{S}_n$ , together with its adjoint  $\mathbf{G}_n$  of its local symmetries, constitute a principal  $\mathbf{G}_n$ -finsheaf of qausets and their finitary local causal (and quantal) homeomorphisms<sup>106</sup>. We may denote this principal finsheaf either by  $\vec{\mathcal{M}}_n := \mathbf{G}_n(\vec{S}_n)$ , or

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<sup>103</sup> $\mathbf{G}_n$  is a group sheaf with carrier or representation or associated sheaf that of qausets  $\vec{S}_n$ . The proper technical name for  $\mathbf{G}_n$  is ‘principal sheaf with structure group  $\mathcal{L}_n$ ’ although, as we also mentioned in the introduction, we use the latter denomination for the pair  $(\vec{S}_n, \mathbf{G}_n)$  (see below).

<sup>104</sup>This topological interpretation of the primitive ideals of an incidence algebra  $\Omega$  associated with a finitary poset substitute  $F$  in (Sorkin, 1991) as ‘space points’, comes from the Gel’fand ‘spatialization procedure’ used in (Zapatrin, 1998, Breslav *et al.*, 1999), whereby, the point-vertices of the poset substitute  $F$  of  $X$  were corresponded to elements of the primitive spectrum  $\mathcal{S}$  of its associated incidence algebra  $\Omega$  which, in turn, are the kernels of the irreducible representations of  $\Omega(F)$  (see previous section). In our causal version  $\vec{\Omega}$  of  $\Omega$ , the primitive spectrum of the former is denoted by  $\vec{\mathcal{S}}$  and its points (*ie*, the primitive ideals of  $\vec{\Omega}_n$ ) are interpreted as ‘coarse spacetime events’ (*ie*, they are equivalence classes of  $X$ ’s point-events relative to our pragmatic observations  $\mathcal{U}_n$  of them of ‘limited power of resolution’) (Raptis and Zapatrin, 2000, Raptis, 2000a).

<sup>105</sup>Hence  $\mathbf{G}_n$  may be thought of as the finitary substitute of the continuous Lie group-manifold  $L^+$  which, due to the (local) quantal character of the qausets in  $\vec{\Omega}_n$ , also inherits some of the latter’s ‘quantumness’ in the sense that since qausets coherently superpose with each other locally according to the FLSP of section 2, so will their symmetry transformations. This is in accord with Finkelstein’s insight that if spacetime is to be regarded as being fundamentally a quantum system, then so must be its structure symmetries (Finkelstein, 1996). See also discussion in  $(b')$  of section 6.

<sup>106</sup>Which may be algebraically represented by the group of incidence algebra homomorphisms as we mentioned in section 2. If, as we noted above, we allow for coherent quantum superpositions between these local quantum causal symmetries as we allow for the causal connections themselves in  $\vec{\Omega}_n$ , then presumably this group is a ‘quantum group’ in a sense akin to (Finkelstein, 1996). Interestingly enough, the Lie algebra  $sl(2, \mathbb{C})$  has been shown (Selesnick, 1994) to result from the quantization of the classical binary alternative *bf2*-the local symmetry of the dyadic cell of the quantum causal net in (Finkelstein, 1988). However, we are not going to explore further the ‘quantum group’ (*ie*, the quantal) character of the finitary local causal symmetries in the group finsheaf  $\mathbf{G}_n$ .

more analytically by  $\vec{\mathcal{M}}_n := (\vec{F}_n, \vec{\Omega}_n, \vec{\mathcal{L}}_n)^{107}$ , with the corresponding local homeomorphisms defining them as finsheaves (with a causal topological interpretation) being denoted as  $\vec{s}_n : (\vec{F}_n, \vec{*}) \rightarrow (\vec{\Omega}_n, \vec{\rho})$  and  $\vec{\lambda}_n : (\vec{\Omega}_n, \vec{\rho}) \rightarrow (\vec{S}_n, \mathbf{g}_n)$ ; where the reticular local causal homeomorphism  $\vec{\lambda}_n$  corresponds a  $\vec{\rho}$ -preserving map to an element in the reticular Lie algebra  $\mathbf{g}_n$  of the structure group  $\mathbf{G}_n = L_n^+$  of the  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n^{108}$ .

The main conjecture in this paper, briefly mentioned at the end of (Raptis, 2000a) and in the introduction, and not to be analytically proved here, is that an inverse system  $\mathcal{K}$  of the  $\mathbf{G}_n$ -finsheaves of qausets  $\vec{\mathcal{M}}_n$  converges to the classical flat Minkowskian  $\mathbf{G}$ -sheaf  $(X \subset \mathcal{M}, \Omega, D, L^+)$ , where  $X$  is a bounded region in the smooth, flat Minkowski manifold  $\mathcal{M}$ , which serves as the base space for the sheaf of smooth differential forms  $\Omega$  on it. This sheaf has as stalks over  $X$ 's point-events isomorphic copies of the  $\mathbb{Z}$ -graded module of Cartan exterior differential forms  $\Omega := \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \dots$ ,  $D$  is the nilpotent and flat Kähler-Cartan connection on the sheaf<sup>109</sup>, while  $L^+$  is the structure group of the sheaf consisting of the global orthochronous Lorentz transformations of  $\mathcal{M}^{110}$ . Heuristic arguments that support this conjecture are:

(a) The topological (*ie*,  $C^0$ ) structure of  $(X \text{ of } \mathcal{M})$  as a topological manifold arises as the limit space of an inverse system of finitary incidence algebras  $\Omega_n(F_n)$ , now topologically interpreted, as shown in (Sorkin, 1991, Raptis and Zapatrin, 2000). It can also be determined from the causally interpreted incidence algebras  $\vec{\Omega}_n(\vec{F}_n)$  as the quote from Bombelli *et al.* (1987) given in the introduction suggests.

(b) The differential (*ie*,  $C^\infty$ -smooth) structure of  $(X \text{ in } \mathcal{M})$  as a differential manifold supporting fibers of modules  $\Omega$  of Cartan's exterior forms, arises as the limit space of an inverse system of finitary incidence algebras  $\Omega_n(F_n)$ , since the latter have been seen to be discrete differential manifolds in the sense of Dimakis and Müller-Hoissen (1999) (Raptis and Zapatrin, 2000, Zapatrin 2000). In fact, as Dimakis *et al.*'s paper (1995) shows, the discrete differential structure of such discrete differential manifolds

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<sup>107</sup>The symbol ' $\vec{\mathcal{M}}_n$ ' for ' $\mathbf{G}_n(\vec{S}_n)$ ' will be explained shortly.

<sup>108</sup>That is, the reticular version of the Lie algebra  $\ell^+$  of the orthochronous Lorentz group  $L^+ = SO(1, 3)^\dagger$  whose algebraic structure is supposed to respect the 'horizontal reticular causal topology' of  $\vec{\Omega}_n$  which is generated by  $\vec{\rho}$  'the germ of the local quantum causal topology' of the qauset stalks  $\vec{\Omega}_n$  of  $\mathbf{G}_n$ 's associated finsheaf  $\vec{S}_n$  (Raptis, 2000a).

<sup>109</sup>As we mentioned in the introduction,  $D$  effects the following (sub)sheaf morphisms in the differential triad  $(X, \Omega, D)$ ;  $D : \Omega^i \rightarrow \Omega^{i+1}$  (Mallios, 1998).

<sup>110</sup>This description of the sheaf  $(\mathcal{M}, \Omega, D, L^+)$  makes it the  $\mathbf{G}$ -sheaf theoretic analogue of a  $\mathbf{G}$ -bundle of exterior forms having as base space the flat Minkowski differential manifold  $\mathcal{M}$ , as fibers modules of smooth Cartan forms on  $\mathcal{M}$ , as flat generalized differential (*ie*, connection) structure the nilpotent Kähler-Cartan differential  $D$ , and as structure group the orthochronous Lorentz group  $L^+$ . One may regard this sheaf as the mathematical structure in which classical as well as quantum field theories (excluding gravity) are formulated.



also determines their (finitary) topology<sup>111</sup>. The differential structure of  $\mathcal{M}$  can also be determined from the causally interpreted incidence algebras  $\vec{\Omega}_n(\vec{F}_n)$  as the quote from Bombelli *et al.* (1987) given in the introduction suggests.

(c) The Minkowski metric  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$  on  $\mathcal{M}$  is determined by the causal incidence algebra  $\vec{\Omega}_n$  associated with the causal set  $\vec{F}_n$  as the quote from (Bombelli *et al.*, 1987) also suggests<sup>112</sup>. It must be emphasized however that in order to determine an indefinite Lorentzian spacetime metric such as  $\eta_{\mu\nu}$ , the causally interpreted finitary incidence algebras must be used, not the topological ones. This is because, as it was shown in (Dimakis and Müller-Hoissen, 1999), the discrete metric  $g$  that is naturally defined on a discrete differential manifold such as the finitary topological incidence algebra of (Raptis and Zapatin, 2000), is positive definite (Riemannian), rather than indefinite (pseudo-Riemannian, Lorentzian). This is the ‘signature problem’ alluded to in the introduction. The solution of the ‘signature problem’ by using causets instead of topological posets justifies Finkelstein’s (1988) and Sorkin’s (1995) demand for a physical causal or temporal topology instead of an unphysical spatial one, as we discussed in the previous section.

(d) The Kähler-Cartan differential operator  $D$  that defines the differential structure of  $\mathcal{M}$  in (b) is a flat connection on the differential triad sheaf  $(\mathcal{M}, \Omega, D)$  (Mallios, 1998), as it is expected to be for the flat Minkowski base space  $\mathcal{M}$ . In (Dimakis and Müller-Hoissen, 1999), a connection  $\nabla$  and its associated curvature  $R := -\nabla^2$  are defined, and compatibility conditions between  $\nabla$  and the definite metric  $g$  are given that make the connection a metric one<sup>113</sup>. However, since as it was mentioned in (c),  $g$  is a positive definite metric,  $\nabla$  will not do, for we are looking for a pseudo-Riemannian (Lorentzian) connection on our finsheaves of causets. Furthermore, as it was also shown in (Dimakis and Müller-Hoissen, 1999), for the most general (universal) discrete differential calculus on a discrete differential manifold,  $\nabla$  reduces to the flat (because nilpotent) Kähler-Cartan differential  $D$ , so that there is no discrete (not even positive definite-Riemannian) ‘gravity’ on it. This is the ‘flatness problem’ alluded to in the introduction. The flatness problem will be tackled in the next section by a straightforward localization or gauging of causets in their finsheaves.

(e) Finally, for the sheaf of global orthochronous Lorentz transformations that we expect to arise as the group sheaf (Mallios, 1998) of (global) symmetries of its

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<sup>111</sup>That is to say, “*differentiability implies continuity*”-the classical motto in mathematical analysis.

<sup>112</sup>As we mentioned in the introduction, the work of Robb (1914) already shows that causality as a partial order determines a Lorentzian metric up to its determinant (spacetime volume-measure). See also (Bombelli and Meyer, 1989, Sorkin, 1990).

<sup>113</sup>That is to say, a connection satisfying  $\nabla g = 0$ .

adjoint<sup>114</sup> flat Minkowskian sheaf  $(\mathcal{M}, \Omega, D)$  from an inverse system of finsheaves  $\mathbf{G}_n = \mathcal{L}_n(\vec{\Omega}_n)$  in the same way that the flat differential triad  $(\mathcal{M}, \Omega, D)$  arises from an inverse system of the finsheaves  $\vec{S}_n = \vec{\Omega}_n(\vec{F}_n)$ , the work of Zeeman (1964) provides significant clues. The key idea from (Zeeman, 1964) for our finitary considerations here is that when causality is modeled after a partial order between events in  $\mathcal{M}$ , its causal automorphisms (*ie*, one-to-one, order-preserving maps of  $\mathcal{M}$ ) constitute a group  $\mathbf{G}$  isomorphic to the orthochronous Lorentz group  $L^+$ . Also,  $\mathbf{G}$  is, by definition, the group of homeomorphisms of  $\mathcal{M}$  regarded as a causal space having for topology the causal Alexandrov (1956, 1967) one (Torretti, 1981). It follows that the maps in the finsheaf  $\mathbf{G}_n$ , being by definition local homeomorphisms of the qauset  $\vec{\Omega}_n$ , respect the local (quantum) causal topology of  $\vec{\Omega}_n$  which, in turn, effectively corresponds to the generating relation  $\vec{\rho}$ . These are the finitary (and quantal) analogues of the causal automorphisms in (Zeeman, 1964), as we argued earlier. In fact, in the next section, by a heuristic implementation of the FEP, FLP and FLRP given in section 2, we will use these finitary causal morphisms to define a finitary, quantal and causal gauge theoretic version of Lorentzian gravity on the gauged  $\vec{\mathcal{M}}_n$ <sup>115</sup>. For the time being we note that the expected Minkowskian classical limit  $\mathbf{G}$ -sheaf  $(X \subset \mathcal{M}, D, \Omega, L^+)$ , being flat, admits of global sections (Mallios, 1998), a result which in physical parlance is known by the following fact: ‘there is a global inertial coordinate patch (frame or gauge) covering flat Minkowski space’ (Torretti, 1981). However, in a curved spacetime  $M$ , there are only local inertial frames (gauges) ‘covering’ (*ie*, with origin) its point-events by the CEP. These are independent (of each other) kinematical frames (gauge possibilities) as we said in section 2 and this ‘kinematical independence’ or ‘gauge freedom’<sup>116</sup> motivates us here to define a non-flat connection on (*ie*, to gauge) the flat  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n$ . Then, the resulting gauged, hence curved, finsheaf will not admit global sections<sup>117</sup> (Mallios, 1998).

We close this section by commenting on the operational significance of our  $\mathbf{G}_n$ -finsheaf model of quantum causality and its (global) causal symmetries. If we take seriously the conjecture above about the convergence of the system  $\mathcal{K} := \{\vec{\mathcal{M}}_n\}$  to the classical flat Minkowskian  $\mathbf{G}$ -sheaf  $(X \subset \mathcal{M}, \Omega, D, L^+)$  at maximum resolution of  $X$  into its point-events *à la* Sorkin (1991) and Raptis (2000*a*), then sound operational

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<sup>114</sup>Or ‘associated’.

<sup>115</sup>That is to say, the principal finsheaf  $\vec{\mathcal{M}}_n$  will be supplied with a non-flat  $\mathbf{g}_n$ -valued spin-Lorentzian connection  $\mathcal{A}_n$  and its associated curvature  $\mathcal{F}_n$ .

<sup>116</sup>To be explained in the next section.

<sup>117</sup>Of particular interest is that it will not admit a global Lorentz-valued connection section  $\mathcal{A}_n$  of its  $\vec{\Omega}_n^1$  sub-sheaf.

meaning may be given to qausets and their finitary symmetries in complete analogy to the one given to topological poset substitutes  $F_n$  of bounded regions  $X$  of continuous spacetime manifolds  $M$  in (Sorkin, 1991, 1995) and their quantal algebraic relatives  $\Omega_n(F_n)$  in (Raptis and Zapatrin, 2000). Since the  $F_n$ s were seen to converge to  $X$ , they were taken to be sound approximations of its point-events, whereby a coarse determination of the locus of an event  $x$  in  $X$  is modeled by an open set about it. We emulate this semantic model for the  $F_n$ s in the case of our qausets  $\vec{\Omega}_n$  as follows: we introduce a new ‘observable’<sup>118</sup> for spacetime events called ‘causal potential (or propensity) relative to our locally finite (coarse) observations  $\mathcal{U}_n$  of them’, and symbolized by  $\vec{\phi}_n$ , so that the causal relation  $x \rightarrow y$  between two events in  $\vec{F}_n$  can be read as ‘ $x$  has higher causal potential than  $y$ ’ (ie, formally:  $\vec{\phi}_n(x) > \vec{\phi}_n(y)$ )<sup>119</sup>. Thus, causality may be conceived as ‘causal potential difference between events relative to our observations of them’<sup>120</sup>.

This definition of  $\vec{\phi}_n$  applies in case the causal set  $\vec{F}_n$  is the causally interpreted finitary poset substitute  $F_n$  of a bounded spacetime region  $X$  as defined in (Raptis, 2000b). If  $F_n$  derives from the locally finite open cover  $\mathcal{U}_n$  of  $X$ ,  $\vec{\phi}_n$  in  $\vec{F}_n$  may be read as follows: the causal potential  $\vec{\phi}_n$  of an event  $x$  in  $X$  relative to our observations  $\mathcal{U}_n$  of  $X$ <sup>121</sup> corresponds to the ‘nerve’  $\mathcal{N}$  covering  $x$  relative to  $\mathcal{U}_n$ , whereby  $\mathcal{N}(x) := \{U \in \mathcal{U}_n \mid x \in U\}$  (Raptis and Zapatrin, 2000)<sup>122</sup>. Then, at the level of resolution of the

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<sup>118</sup>Actually, to be established as a dynamical variable in the next section where we gauge  $\vec{\mathcal{M}}_n$ .

<sup>119</sup>This formal labeling of events by  $\vec{\phi}_n$  is in complete analogy to the natural number  $\mathbb{N}$ -labeling of events *à la* Rideout and Sorkin (2000). There the sequential growth dynamics proposed for causets was seen to be independent from their  $\mathbb{N}$ -labeling thus in some sense ‘gauge independent’ of an external (background) discrete  $\mathbb{N}$ -valued time gauge. In the next section we will see that similarly the reticular gauge connection  $\mathcal{D}_n$  based on which the dynamical law for qausets will be formulated as an equation between sheaf morphisms, will be seen to be gauge  $\mathcal{U}_n$ -independent, thus also  $\mathcal{A}_n$ -covariant. See also (Raptis, 2000e) for more on this.

<sup>120</sup>In a plausible ‘particle interpretation’ of our reticular scheme, whereby a network of causet (or qauset) connections is interpreted in the manner of Dimakis *et al.* (1995) as the reticular pattern of the dynamics of particles (or quanta) of causality which may be called ‘causons’ for obvious reason, the causal connection  $x \rightarrow y$  has the following rather natural physical interpretation in terms of the causal potential  $\vec{\phi}_n$ : ‘a causon descends from the event  $x$  of higher causal potential to the event  $y$  of lower causal potential’. This is in literal analogy, for instance, with the motion of an electron in an electromagnetic potential gradient, hence the natural denomination of  $\vec{\phi}_n$  as ‘causal potential’.

<sup>121</sup>One can equivalently call it ‘the causal potential of an event  $x$  in  $X$  at the limit of resolution of  $X$  corresponding to  $\mathcal{U}_n$ ’ (Raptis, 2000a). The definition of  $\vec{\phi}_n$  as being ‘relative to our coarse spacetime observations’ is reflected by its index which is the same as that of the locally finite open cover  $\mathcal{U}_n$  of  $X$ -a finite  $n$  signifying a pragmatic limited (finite), but at the same time coarse and perturbing, power of resolution of  $X$  into its point-events.

<sup>122</sup>As we also saw in the previous section, in (Raptis and Zapatrin, 2000) nerves were seen to

spacetime manifold-now regarded as a causal space, corresponding to  $\vec{F}_n$ ,  $x \rightarrow y$  (ie, ‘ $x$  causes  $y$ ’) means operationally that  $\mathcal{N}(y) \subset \mathcal{N}(x)$  (ie, ‘every (rough) observation of  $y$  is a (coarse) observation of  $x$ ’)<sup>123</sup>; thence,  $\vec{\phi}_n(x) > \vec{\phi}_n(y)$ . In terms of the definition of the smallest open sets in  $\mathcal{U}_n$  containing  $x$  and  $y$ ,  $\Lambda(x)$  and  $\Lambda(y)$ , given in section 3 and in (Sorkin, 1991, Raptis, 2000b), that is to say,  $\Lambda(x) := \bigcap \{U \in \mathcal{U}_n \mid x \in U\} \equiv \bigcap \mathcal{N}(x)$ ,  $\mathcal{N}(y) \subset \mathcal{N}(x)$  reads  $\Lambda(x) \subset \Lambda(y)$  with ‘ $\subset$ ’ standing for strict set theoretic inclusion<sup>124</sup>. This is precisely how the topological partial order  $\rightarrow$  in  $F_n$  was defined in (Sorkin, 1991), only in our  $\vec{F}_n$  it is re-interpreted causally (Raptis, 2000b)<sup>125</sup>. It must be mentioned that such a conception of (quantum) causality as a difference in cardinality (or degree) was first conceived in a different mathematical model by Finkelstein (1969), while in (Breslav *et al.*, 1999), and in a model similar to ours, the collection  $\mathcal{U}_n$  of open sets were assigned to teams (or organizations) of ‘coarse observers’ of spacetime topology and it is explicitly mentioned that the relation  $x \rightarrow y$  means that “*the event  $x$  has been observed more times (by the team) than the event  $y$* ”. However there, the ‘ $\rightarrow$ ’ obtained from Sorkin’s ‘equivalence algorithm’ (section 3) is seen to still have its original topological meaning and is not given a directly causal significance like in our scheme<sup>126</sup>.

From the definition of  $\vec{\phi}_n$  above, it follows that the generator of local (ie, contiguous<sup>127</sup>) causal potential differences between events in  $\vec{F}_n$  corresponds to the relation of immediate causality  $\xrightarrow{*}$  linking events, say  $x$  and  $y$ , such that  $\Delta\vec{\phi}_n(x, y) := \vec{\phi}_n(x) - \vec{\phi}_n(y) = 1$ . That is to say, we may symbolize this ‘contiguous’ causal potential difference-the ‘germ’ of the quantum causal potential, by  $\vec{\phi}_n^*$ . If we pass to the causet  $\vec{\Omega}_n$  associated with  $\vec{F}_n$ , or equivalently, to the finsheaf  $\vec{S}_n$  of causets, the aforementioned generator of causal potential differences assumes a completely algebraic expression as  $\vec{\rho}$ . Again, we recall from section 3 that  $I_p \vec{\rho} I_q \Leftrightarrow I_p I_q (\neq I_q I_p) \subsetneq I_p \cap I_q$

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be simplicial complexes and the topological discretization of manifolds based on them is due to Alexandrov (1956, 1961). In (Raptis and Zapatrin, 2000), the degree of a nerve of a covering is its cardinality, namely, the number of the open sets in the covering that constitute it.

<sup>123</sup>See (Breslav *et al.*, 1999) for a similar operational semantics, but applied to the topological not to the causal structure of spacetime like we do here.

<sup>124</sup>Simplicially speaking, ‘ $x$  is a face of  $y$  with respect to  $\mathcal{U}_n$ ’. See section 3 and (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000).

<sup>125</sup>Note that event-vertices in the causet  $\vec{F}_n$  that are causally unrelated (ie, in some sense ‘space-like’) are covered by different nerves in  $\mathcal{U}_n$  of equal degree or cardinality (Raptis and Zapatrin, 2000).

<sup>126</sup>That is, the quantum observable or dynamical variable in their theory is topology, not local causality.

<sup>127</sup>See section 2.

generates the quantum causal Rota topology of  $\vec{S}_n$  by relating primitive ideals  $I_p$  and  $I_q$  in  $\vec{S}(\vec{\Omega}_n)$  ( $p, q \in \vec{F}_n$ ) if and only if  $p \rightarrow q$  and  $\nexists r \in \vec{F}_n : p \rightarrow r \rightarrow q$  (ie, iff  $p \xrightarrow{*} q$  in  $\vec{F}_n$ ) (Breslav *et al.*, 1999, Raptis, 2000b).

Here, in the algebraic setting of gausets, the generator of quantum causality (ie, the germ  $\vec{\phi}_n^*$  of the quantum causal potential  $\vec{\phi}_n$ ) relative to our finitary spacetime observations in  $\mathcal{U}_n$ ,  $\vec{\rho}$ , has the following operational and quantal *à la* Heisenberg (because non-commutative algebraic) meaning that reads from its very algebraic definition: point-events in  $\vec{\Omega}_n$ , which correspond to primitive ideals in  $\vec{S}(\vec{\Omega}_n)$ <sup>128</sup>, have a product ideal that is strictly included in their intersection ideal, with the ‘directedness’ (asymmetry) of their immediate quantum causal connection, say ‘from- $p$ -to- $q$ ’ ( $p \xrightarrow{*} q$ ), being reflected in the non-commutativity of their corresponding ideals in  $\vec{\Omega}_n$  (ie,  $I_p I_q \neq I_q I_p$ )<sup>129</sup>. This operational description of quantum causality in  $\vec{\Omega}_n$  relative

<sup>128</sup>Recall from (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000b) and section 3 the definition of the primitive ideals in the corresponding quantum topological  $\Omega_n(F_n)$ :  $I_p := \text{span}\{|q \succ r| : |q \succ r| \neq |p \succ r| \}$ ; where  $|q \succ r| := (q \rightarrow r) \in F_n$ . Parenthetically, it is rather interesting to observe in this definition of the primitive ideals (points) in the quantal topological incidence algebras  $\Omega_n(F_n)$  that the elements (ket-bras) that constitute them are quantal acts of determination of what in the classical limit space will emerge as ‘momentum (covector) states’ and serial concatenations thereof (ie, ‘spacetime time-like paths’); see physical interpretation of the  $\Omega^i$ s ( $i \geq 1$ ) in (Raptis and Zapatrin, 2000). By this very definition of the  $I_p$ s in  $\mathcal{S}(\Omega_n(F_n))$ , we see that the operations of determination of pure quantum spacetime states (events) in  $\Omega_n$ , namely, the elements of  $\Omega^0$  (the  $|p \succ r|$ s in the definition of the  $I_p$ s above; Raptis and Zapatrin, 2000), are excluded from them. So, operations of determination of what classically (ie, at the non-pragmatic decoherence limit of infinite refinement of the spacetime continuum into its point-events) appear as momentum states tangent to spacetime ‘position states’ (point-events) are ‘incompatible’ or ‘complementary’ in Bohr’s sense with (ie, they exclude) quantum acts of localization of the latter. This observation shows that some kind of quantum uncertainty is built into our Rota algebraic scheme *ab initio* thus it further justifies the physical interpretation of the limit of infinite localization of spacetime events as Bohr’s correspondence principle (Raptis and Zapatrin, 2000). The quantum character of the non-commutative topology generated by the local (and dynamical) quantum causality  $\vec{\rho}$  is analytically studied in (Raptis, 2000e).

<sup>129</sup>This is a first indication of a fundamental non-commutativity of (acts of localization of) ‘points’ (ie, ‘coarse spacetime events’) underlying quantum causal topology in a model like ours (where ‘points’ are represented by primitive ideals in the primitive spectra  $\vec{S}$  of the incidence algebras  $\vec{\Omega}$  involved). In a coming paper (Raptis, 2000e), the incidence algebras modeling gausets here, as well as their localizations, are studied in the light of scheme theory (Hartshorne, 1983, Shafarevich, 1994) and a non-commutative dynamical local quantum causal topology for (at least the kinematics of) Lorentzian quantum gravity is defined based on such non-abelian schematic algebra localizations in much the same way to how Non-Commutative Algebraic Geometry was defined in (Van Oystaeyen and Verschoren, 1981) based on non-abelian Polynomial Identity (PI) ring localizations-it being understood that Rota algebras can be regarded as PI rings (Freddy Van Oystaeyen in private communi-

to our coarse observations of events in a bounded region of spacetime-now interpreted as a causal space, follows from the operational description of causality in the causet  $\vec{F}_n$  from which it derives via the causal potential ‘observable’  $\vec{\phi}_n$  defined above.

All in all, (quantum) causality is operationally defined and interpreted as a ‘power relationship’ between spacetime events relative to our coarse observations (or approximate operations of local determination) of them, namely, if events  $x$  and  $y$  are coarsely determined by  $\mathcal{N}(x)$  and  $\mathcal{N}(y)$  with respect to  $\mathcal{U}_n$ , and  $\mathcal{N}(y) \subset \mathcal{N}(x)$ , then ‘ $x$  causes  $y$ ’. The attractive feature of such a definition and interpretation of causality is that, by making it relative to  $\mathcal{U}_n$ , we render it ‘frame- or observation-dependent’<sup>130</sup>, ultimately, relativistic<sup>131</sup>.

In the same way, one can give operational meaning to the finitary local (quantum) causal automorphisms of the  $\vec{\Omega}_n$ s in  $\vec{S}_n$  mentioned above. They represent finitary operations of ‘approximation’<sup>132</sup> of the local symmetries of quantum causality as encoded in the finsheaf  $\vec{S}_n$  and they too are organized in the finsheaf  $\mathbf{G}_n$ . The operational

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cation). It must be a fruitful project to compare the resulting ‘non-commutative topology for curved quantum causality’ in (Raptis, 2000e) with the one defined and studied in (Van Oystaeyen, 2000a). The second author (IR) wishes to thank Freddy Van Oystaeyen for motivating such a study in a crucial private communication and in two research seminars; see (Van Oystaeyen, 2000b). Ultimately, the deep connection for physics is anticipated to be one between such a non-commutative conception of the local quantum causal topology of spacetime and the fundamental micro-local quantum time asymmetry expected of ‘the true quantum gravity’ (Penrose, 1987). Again, such a fundamental time asymmetry in a curved finitistic quantum causal topological space similar to ours has already been anticipated by Finkelstein (1988).

<sup>130</sup>One may think of the open  $U$ s in  $\mathcal{U}_n$  as some sort of ‘rough coordinate patches’ or ‘coarse frames’ or even as ‘fat gauges’ (Mallios, 1998) ‘covering’ or coarsely measuring (approximately localizing) the point-events in  $X$ .

<sup>131</sup>Recall that the causal potential  $\vec{\phi}_n$  of events is defined relative to our coarse observations  $\mathcal{U}_n$  of them, so that, as we will see in the next section, its localization (gauging) and relativization will effectively amount to establishing a local transformation theory for it that respects its dynamics (due to a finitary sort of Lorentzian quantum gravity), in the sense that this dynamics becomes independent of the level of resolution corresponding to our observations  $\mathcal{U}_n$  of spacetime into its events, or equivalently, it becomes independent of the local gauges (frames)  $\mathcal{U}_n$  that one lays out to chart the spacetime events and measure, albeit coarsely, physical attributes such as the gravitational field ‘located there’ (Mallios, 1998). This will be then the transcription of the fundamental principle of GR, which requires that the laws of physics are invariant under the diffeomorphism group of the smooth spacetime manifold  $\text{Diff}(M)$  (the principle of General Covariance), in a sheaf theoretic model for a curved finitary quantum causal space: ‘the laws of physics are equations expressed in terms of sheaf morphisms’-the main sheaf morphism being the connection  $\mathcal{D}$  (Mallios, 1998). We will return to this principle in section 5 where we define  $\mathcal{D}_n$  as a finsheaf morphism in our scheme and further discuss its quantum physical implications in section 6.

<sup>132</sup>The inverted commas for the word ‘approximation’ were explained in the introduction.

interpretation of the elements of  $\mathbf{G}_n$  as coarse reticular (and quantal) replacements of the continuous local orthochronous Lorentz Lie symmetries of the smooth gravitational spacetime of GR will become transparent in the next section when we gauge the flat Minkowskian  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n$  by providing a non-flat  $\mathfrak{g}_n$ -valued connection 1-form  $\mathcal{A}_n$  on it.

## 5. GAUGING QUANTUM MINKOWSKI SPACE: NON-FLAT CONNECTION ON $\vec{\mathcal{S}}_n$

The reader was prepared in the previous sections for the present one where we will attempt to curve the flat and quantal  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n := \mathbf{G}_n(\vec{\mathcal{S}}_n)$  by gauging or localizing it. As it was mentioned earlier, this procedure is tantamount to defining a reticular non-flat spin-Lorentzian connection  $\mathcal{A}_n$ <sup>133</sup> that takes values in the Lie algebra  $\mathfrak{g}_n$  of the group finsheaf  $\mathbf{G}_n$  adjoint to  $\vec{\mathcal{S}}_n$  consisting of the latter's local quantum causal symmetries—the finitary and quantal substitute of the continuous orthochronous Lorentz Lie group manifold  $L^+$  which, in turn, is the structure group of (global symmetries of) the flat  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n$ <sup>134</sup>. The resulting curved  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{P}}_n := (\vec{F}_n, \vec{\Omega}_n, \mathcal{L}_n, \mathcal{D}_n := D_n + \mathcal{A}_n)$  may be regarded as a finitary, causal and quantal replacement of the classical structure  $\mathcal{P}$  on which GR is formulated as a gauge theory of a spin-Lorentzian connection 1-form  $\mathcal{A}$ . This model  $\mathcal{P}$  of the curved classical spacetime structure of GR, as it was noted in the introduction, is a principal fiber bundle of modules  $\Omega$  of Cartan differential forms, over a region  $X$  of a  $C^\infty$ -smooth Lorentzian spacetime manifold  $M$ , with structure group  $L^+ := SO(1, 3)^\uparrow$  and a non-flat  $so(1, 3)^\uparrow$ -valued gravitational gauge connection  $\mathcal{A}$  on it. As it was also briefly noted in the

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<sup>133</sup>The reader should note the index  $n$  given to the connection  $\mathcal{A}$  that is the same as the one given to the causet  $\vec{F}_n$ , its associated qauset  $\vec{\Omega}_n$  and the latter's local quantum causal symmetries  $\mathcal{L}_n$ . Properly viewed, the connection  $\mathcal{A}$  on the  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n := (\vec{F}_n, \vec{\Omega}_n, D_n, \mathcal{L}_n)$  in focus inherits the latter's 'finite degree of resolution'  $n$  of the region  $X$  of the curved spacetime manifold  $M$  by our coarse and dynamically perturbing observations  $\mathcal{U}_n$  of its events, their causal ties and the symmetries of the latter (Raptis, 2000a). The reader should notice that the index  $n$  is also given to the reticular Kähler-Cartan differential  $D$  in  $\vec{\mathcal{M}}_n$  just to remind one of its discrete character *à la* Dimakis and Müller-Hoissen (1999).

<sup>134</sup>As we contended in the last two sections and briefly alluded to in the introduction, this is the finitary, causal and quantal substratum underlying the flat principal Minkowskian sheaf  $(X \subset \mathcal{M}, \Omega, D, L^+)$  of modules of Cartan differential forms over the bounded region  $X$  of the Minkowski manifold  $\mathcal{M}$ , equipped with the flat Kähler-Cartan exterior differential (connection)  $D$  and having as continuous structure group  $L^+$  that of (global) orthochronous Lorentz symmetries of  $\mathcal{M}$ . This 'classical' sheaf, being flat, admits of global sections of its adjoint structure group sheaf  $L^+$  over  $\mathcal{M}$  (Mallios, 1998).

introduction, this classical model may be equivalently thought of as a rigorous mathematical formalization of the spinorial formulation of GR due to Bergmann (1957) when, fiber-wise (*ie*, locally), one corresponds the real Minkowski spacetime vectors dual to the Minkowskian co-vectors  $\Omega^1$  in the Cartan bundle, to  $H(2, \mathbb{C})$ -the space of Hermitian bispinors (or Hermitian biquaternions) (Finkelstein, 1996) in the complex completely antisymmetric tensor bundle  $T^* \simeq \Omega_{\mathbb{C}}^1 \simeq S \otimes \tilde{S} \equiv (\tilde{S}^* \otimes S^*)^{*135}$  over the 2-dimensional Grassmannian subspaces  $Gr_2(\mathbb{C}^4)$  of complexified Minkowski space  $\mathbb{C}^{4136}$  (Manin, 1988, Selesnick, 1991). This correspondence is the usual one:  $x_\mu \rightarrow x_\mu \sigma_\mu =: x_{a\dot{b}} \equiv s \otimes s^\sim$ ; ( $s \in S \simeq \mathbb{C}^2, s^\sim \in \tilde{S} \simeq \mathbb{C}^2$ )<sup>137</sup>; with  $\sigma_\mu$  the usual tetrad of Pauli's  $2 \times 2$  spin-matrices.

Then, again fiber-wise, the correspondence between the structure groups of the Bergmann and the Cartan principal fiber bundles is the usual projective one given by the 2-to-1 map:  $\rho : SL(2, \mathbb{C}) \rightarrow SO(1, 3)^\uparrow =: L^+$ , which reflects the fact that  $SL(2, \mathbb{C})$  is the double cover of  $SO(1, 3)^\uparrow$ . However, again as it was briefly mentioned in the introduction, locally in the group fiber (*ie*, Lie algebra-wise), the two structure groups are isomorphic, since  $so(1, 3)^\uparrow \simeq sl(2, \mathbb{C})$ <sup>138</sup>.

Finally, as it was also alluded to in the introduction, our holding that the curved  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{P}}_n$  is a finitary, causal and quantal replacement of the classical Cartan-Bergmann  $\mathbf{G}$ -bundle  $\mathcal{P} = (X, \Omega, L^+, \mathcal{D} := D + \mathcal{A})$  model of GR, basically rests on

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<sup>135</sup>Where ‘ $*$ ’ denotes ‘dual space’ and  $T \simeq (\tilde{S}^* \otimes S^*)$  is fiber-wise isomorphic to complex Minkowski space  $\mathbb{C}^4$  (Selesnick, 1991, 1994, 1998).

<sup>136</sup>The spinorial formulation of GR, as well as the self-dual gravity of Ashtekar (Ashtekar, 1986, Baez and Muniain, 1994), inevitably involve complexified spacetime. Imposing reality conditions or recovering real spacetime ( $\mathbb{R}^4$ ) from a complex model (Finkelstein, 1988, Selesnick, 1994, Baez and Muniain, 1994) is certainly not an easy business or without physical significance. Likewise, being incidence algebras, our qausets may be regarded as vector spaces over the field  $\mathbb{C}$ , so, if anything, the finitary and quantal version of  $\mathcal{M}$  as a causal space presented in sections 3 and 4 is surely complex in case we use  $\mathbb{C}$  for linear coefficients (amplitudes) over which qausets superpose coherently in the  $\vec{\Omega}_n$  stalks of  $\vec{\mathcal{M}}_n$  (Raptis, 2000b). However, we are not going to present here the transition from complex to real spacetime and gravity. On the other hand, we must be careful not to infuse *ab initio* the real or the complex number continua into our inherently reticular scenario, for then we will be begging the question ‘discrete before continuous?’. Which amplitude (c-coefficient) structure one must use for qausets and other algebraic structures that have been suggested to model quantum spacetime and gravity is still an open problem (Chris Isham in private correspondence).

<sup>137</sup> $S$  is the space of spinors transforming under the fundamental irreducible representation of  $SL(2, \mathbb{C})$ , while  $\tilde{S}$  consists of the conjugate spinors that transform as vectors under the conjugate (and inequivalent to the fundamental) irreducible representation of  $SL(2, \mathbb{C})$ . See (Selesnick, 1991, 1994, 1998) for notation and relevant definitions.

<sup>138</sup>Hence our calling  $\mathcal{P}$  ‘the Cartan-Bergmann  $\mathbf{G}$ -bundle’ and  $\mathcal{A}$  on it ‘the spin-Lorentzian connection’.



the idea that an inverse system of the former curved finsheaves of qausets yields, at the operationally ideal limit of finest resolution or localization of  $X$  into its point-events, their causal ties and the local symmetries thereof, which limit, in turn, may be interpreted as Bohr's correspondence principle (Raptis and Zapatrin, 2000), the latter as a classical gravitational spacetime structure (Raptis, 2000a).

Thus, we consider a bounded region  $X$  of a curved smooth spacetime manifold  $M$ . We assume that gravity is represented by a non-flat  $sl(2, \mathbb{C})$ -valued connection 1-form  $\mathcal{A}$  on the curved Cartan-Bergmann  $\mathbf{G}$ -bundle  $\mathcal{P} = (X, \Omega, L^+, \mathcal{D} = D + \mathcal{A})$ . First we discuss a mathematical technicality that our finsheaf theoretic model should meet in order to be able to define a (non-flat) connection  $\mathcal{D}_n$ <sup>139</sup> on the (flat) finsheaf  $\vec{\mathcal{M}}_n$ . Two sufficient conditions for the existence of a connection  $\mathcal{D}$  on an algebra or vector sheaf or bundle over a manifold  $M$ , regarded as a topological space, are that  $M$  is paracompact and Hausdorff<sup>140</sup> (Mallios, 1998). It is expected that, since our finsheaves of qausets are finitary (and quantal) replacements of an at least  $T_1$ <sup>141</sup> and relatively compact<sup>142</sup> topological space  $X$  (Sorkin, 1991, Raptis, 2000a), if we relax  $T_2$  to  $T_1$  and paracompactness to relative compactness, we are still able to define a connection  $\mathcal{D}_n$  on a vector or algebra sheaf over it such as our  $\vec{\mathcal{M}}_n$  (Raptis, 2000a). So  $\mathcal{D}_n$  exists (*ie*, is 'defineable') on  $\vec{\mathcal{M}}_n$ . In fact, we know that since  $D_n$  is already

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<sup>139</sup>Note that until now we used the gauge potential  $\mathcal{A}$  for the mathematical concept of connection  $\mathcal{D}$ , when, in fact,  $\mathcal{A}$  is just the part of  $\mathcal{D} = D + \mathcal{A}$  that makes it non-trivial (*ie*, non-flat) (Mallios, 1998). This is the physicist's 'abuse' of the concept of connection, presumably due to his rather 'utilitarian' or at least 'practical' attitude towards mathematics, namely, that he is interested on the part of  $\mathcal{D}$  that is responsible for curvature (which can be physically interpreted as the gauge potential of a physical force). In fact, the substitution  $D \rightarrow \mathcal{D} = D + \mathcal{A}$  is coined 'gauging' in the physics jargon, when  $D$  is from a mathematical point of view a perfectly legitimate connection; albeit, a trivial (*ie*, flat) one (Mallios, 1998). The same 'abuse' of  $\mathcal{D}$  is encountered in (Baez and Muniain, 1994; see chapter 5) where only the gauge potential  $\mathcal{A}$  is coined 'connection'. Here, we too adopt a physicist's approach and by 'gauging our flat Minkowskian principal finsheaf  $\vec{\mathcal{M}}_n$ ' essentially we mean 'adjoining a non-zero connection term  $\mathcal{A}_n$  to its flat differential  $D_n$ '. This asymphony between the mathematician's and the physicist's conception of the notion of connection aside, one should always keep in mind that  $\mathcal{D}$  is a generalized differential operator, with its non-zero part  $\mathcal{A}$  generalizing or extending the usual differential operator  $D$ .

<sup>140</sup>We recall that a topological space  $M$  is said to be paracompact if every open cover of it admits a locally finite refinement. Also,  $M$  is said to be Hausdorff or  $T_2$ , when it satisfies the second axiom of separation of point-set topology which holds that every pair of points of  $M$  have non-intersecting (disjoint) open neighborhoods about them.

<sup>141</sup>We recall that a topological space  $X$  is said to be  $T_1$  if for every pair of points  $x$  and  $y$  in it there exist open neighborhoods  $O_x$  and  $O_y$  containing them such that  $x \notin O_y$  and  $y \notin O_x$ .

<sup>142</sup>As it was also noted earlier, a topological space  $X$  is said to be relatively compact, or bounded in the sense of (Sorkin, 1991), when its closure is compact.

defined on the qauset stalks of  $\vec{\mathcal{M}}_n$  à la Dimakis and Müller-Hoissen (1999)<sup>143</sup> and it is seen to effect sub-sheaf morphisms  $D_n : \vec{\Omega}_n^i \rightarrow \vec{\Omega}_n^{i+1}$  there (Mallios, 1998); albeit, it is a flat connection (Mallios, 1998, Dimakis and Müller-Hoissen, 1999). In turn, this  $\mathcal{D}_n = D_n$  on the finitary, causal and quantal Minkowskian finsheaf  $\vec{\mathcal{M}}_n$  means that  $\mathcal{A}_n = 0$  throughout  $\vec{\mathcal{M}}_n$ , so that by our physical terminology the latter is an ungauged (flat) finsheaf<sup>144</sup>.

To curve the flat finsheaf  $\vec{\mathcal{M}}_n$  by adjoining to its flat connection  $D_n$  a non-zero term  $\mathcal{A}_n$ , we immitate in our finitary context how the curved smooth spacetime manifold  $M$  of GR may be thought of as the result of localizing or gauging the flat Minkowski space  $\mathcal{M}$  of SR. Locally, (*ie*, event-wise), one raises<sup>145</sup> an isomorphic copy of  $\mathcal{M}$  over each spacetime event  $x \in X \subset M$ , thus implementing the CEP of section 2. Hence formally,  $\mathcal{M}$  acquires an event-index  $x$  ( $\forall x \in X$ ),  $\mathcal{M}_x$ , and may be regarded as some kind of ‘fiber space over  $x$ ’<sup>146</sup>. In view of the differential (*ie*,  $C^\infty$ -smooth) character of  $M$ , which, in turn, may be thought of as implementing the CLP discussed in section 2 (Einstein, 1924),  $\mathcal{M}_x$  is geometrically interpreted as ‘the space tangent to  $M$  at  $x$ ’, so that collectively,  $TM := \bigcup_{x \in M} \mathcal{M}_x$  is the locally Minkowskian tangent bundle of  $M$  (Göckeler and Schücker, 1990) having for fibers  $\mathcal{M}_x$  (*ie*, local isomorphs of flat Minkowski space).

Then, the term ‘gauging’ effectively corresponds to regarding these local isomorphs of flat Minkowski space as ‘independent kinematical worlds’, in the sense that two vectors  $v$  and  $v'$  living in the vector spaces  $\mathcal{M}_x$  and  $\mathcal{M}_{x'}$ , respectively, are ‘incomparable’, in that one is not allowed to form linear combinations thereof<sup>147</sup>. Alternatively, one may describe this in a geometrical way by saying that in a gauged space, such as the vector bundle  $TM$  (Göckeler and Schücker, 1990), there is no natural relation of distant parallelism between its fibers. A ‘rule’ that enables one to compare vectors at different fiber spaces, thus it establishes some kind of relation of distant parallelism or, algebraically speaking, ‘distant linear combineability’ in  $TM$ , is provided by the

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<sup>143</sup>See also (Zapatrin, 2000).

<sup>144</sup>As we also mentioned in the previous section, the important point to retain from the discussion above is that in a sheaf theoretic context like ours the role of connection  $\mathcal{D}$  is as a sheaf morphism (Mallios, 1998). We will come back to it shortly when we formulate a finsheaf theoretic version of the Principle of General Covariance of GR in the gauged  $\vec{\mathcal{M}}_n$ .

<sup>145</sup>That is to say, as if it is a ‘vertical structure’.

<sup>146</sup>As we mentioned in section 2,  $\mathcal{M}_x$  may be physically interpreted as a local kinematical (possibility) space for GR implementing the kinematical CEP in the sense that GR’s main dynamical variable,  $g_{\mu\nu}$ , can be locally reduced or ‘gauged away’, by passing to a locally inertial frame, to the flat  $\eta_{\mu\nu}$  of the  $\mathcal{M}$  of SR.

<sup>147</sup>For instance, one is not supposed to be able to compute their difference  $v' - v$  which is the crucial operation for defining the differential operator  $D$  in general.

concept of connection  $\mathcal{D}$  (Mallios, 1998). The geometrical interpretation of  $\mathcal{D}$ , and one which shows an apparent dependence of this concept on the background geometric spacetime manifold  $M$ <sup>148</sup> is as a parallel transporter of vectors along smooth curves in  $M$  joining  $x$  with  $x'$ . Then, curvature  $\mathcal{F}$ , in the classical model for spacetime corresponding to the differential manifold  $M$ , is geometrically conceived as the anholonomy of  $\mathcal{D}$  when the latter parallelly transports vectors along closed smooth curves (loops) in  $M$ —certainly a non-local conception of the action of  $\mathcal{D}$ <sup>149</sup>.

The second problem that we face is one of physical semantics: we want to interpret the non-flat part  $\mathcal{A}_n$  of  $\mathcal{D}_n$  in a finitary causal way. In the classical curved spacetime model  $\mathcal{P}$ ,  $\mathcal{A}$ , apart from its usual interpretation as the gravitational gauge potential, may be physically interpreted as the smooth dynamically variable (field of the) local causal connections between the events of the  $C^\infty$ -smooth spacetime region  $X$ <sup>150</sup>. Since the fibers of the curved  $TM$  (or the Minkowskian covectors in the  $\Omega^1$  sub-bundle of  $\mathcal{P}$ ) are local isomorphs of flat Minkowski space, the action of the spin-Lorentzian gravitational connection 1-form  $\mathcal{A}$  on Minkowski vectors living in  $TM$ 's fibers, besides its geometrical interpretation as ‘parallel translation’ above, may be alternatively interpreted in a causal way as follows: point-wise on the curve along which the vectors are transported, the transitive, inertial Minkowskian causality  $\rightarrow$  is preserved<sup>151</sup>. Equivalently put, if  $x$  is a point in the curve and  $v(x) \in \mathcal{M}_x$  is

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<sup>148</sup>And we say ‘apparent’, because, as we will see shortly, in our scheme  $\mathcal{D}_n := D_n + \mathcal{A}_n$  does not depend essentially on the geometric base spacetime, since it derives locally from the very algebraic structure of the stalks of the finsheaf of gausets (*ie*, from the structure of the quantally and causally interpreted incidence algebras). This is the main lesson we have learned from the Abstract Differential Geometry theory developed in (Mallios, 1998), namely, that  $\mathcal{D}$ , the main object with which one can actually do Differential Geometry, is of an algebraic (*ie*, analytic) nature and does not depend on any sort of ‘ambient geometric space’. For instance, the two global (topological) conditions for the existence of  $\mathcal{D}$  on  $M$  mentioned above, namely, that the latter is a paracompact and Hausdorff topological space, are sufficient, but by no means necessary. Such an independence is welcome from the point of view of both classical and quantum gravity where the spacetime manifold, regarded as an inert geometrical background base space, has shown to us its pathological, ‘unphysical nature’ in the form of singularities and the non-renormalizable infinities that plague the field theories defined on it.

<sup>149</sup>In contradistinction to this classical geometric conception of curvature, in the sense that it depends on the existence of spacetime loops in  $M$  and that it is the effect of the action of  $\mathcal{D}$  as the parallel transporter of geometric entities (smooth tensor fields) along them, we will be able to give shortly a purely local sort of curvature  $\mathcal{F}_n$  stalk-wise in our gauged finsheaves of gausets.

<sup>150</sup>See section 2.

<sup>151</sup>One may conceive in this sense the standard requirement in GR that ‘the connection is compatible with the metric tensor field  $g_{\mu\nu}$ ’, or that ‘ $\mathcal{D}$  is a metric connection’ (*ie*, formally, that  $\mathcal{D}g = 0$ ). This geometrical requirement for ‘zero spacetime distortion’ seems to be a convenience coming from the classical continuum model  $M$  of spacetime (*ie*, in pseudo-Riemannian geometry on  $M$ , the metric  $g_{\mu\nu}$

the value of a vector field  $v$  at  $x$ <sup>152</sup>, the ‘coupling’  $\mathcal{A}(x)[v(x)]$ <sup>153</sup> may be thought of as a local Lorentz transformation (‘infinitesimal spacetime rotation’) of  $v(x)$ ; hence, by definition, it preserves the local causal structure of the curved  $TM$ , namely, the Minkowski lightcone based at (or with origin)  $x$  in  $\mathcal{M}_x$ <sup>154</sup>. In this sense, one may equivalently interpret the gravitational gauge connection  $\mathcal{A}$  as the dynamical field of local causality, as we noted in section 2. We adopt this physical interpretation for our finitary  $\mathcal{A}_n$ .

Now we come to the crucial point of the present paper that was briefly mentioned at the end of section 2 and in the footnote above concerning metric connections. From a causal perspective, like the one we have adopted here, a sheaf may be the ‘natural’ mathematical structure to model such (dynamically variable) local causal connections such as  $\mathcal{A}$ , because of the following rather heuristic argument which in a sense motivated us to study finsheaves of qausets in the first place: by definition, a sheaf is a local homeomorphism (Bredon, 1967, Mallios, 1998, Raptis, 2000a), so that when one is interested in the (dynamically varying) causal topology of spacetime like we are in the present paper where our  $\mathbf{G}_n$ -finsheaf of qausets is supposed to be the ‘quantum discretization’ (Raptis and Zapatrin, 2000) of the local causal topology (*ie*, the causal connections between events) and its local symmetries of a bounded region  $X$  of a curved smooth spacetime manifold  $M$ , a sheaf preserves precisely the (germs of the) local causal topology of the base space<sup>155</sup>. But, in our case, the latter are precisely the immediate causality (contiguity) relations  $\xrightarrow{*}$  in the causal set  $\vec{F}_n$  that are mapped by the sheaf (regarded as a local homeomorphism  $\vec{s}$ <sup>156</sup>) to the germ-relations  $\vec{\rho}$  of the quantum causal Rota topology of the qausets  $\vec{\Omega}_n$ , thus defining

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determines a unique metric connection  $\mathcal{D}$ -the usual Levi-Civita one  $\Gamma$  (Torretti, 1981)), but it may not hold in the quantum deep which assumes no pre-existent  $M$ , let alone a  $g_{\mu\nu}$  on it (Finkelstein, 1996). Like Finkelstein however, only for convenience we assume that at every finite level  $n$  of resolution of spacetime  $\mathcal{A}_n$  is metric-a convenience reflecting the fact that in our finsheaf theoretical model the connection is naturally expected to respect the local (quantum) Minkowskian causal topology  $\vec{\rho}$  of the qauset stalks (see sections 2-4, the discussion in the next paragraph and in section 6).

<sup>152</sup>Technically, a vector field is a cross-section of the vector bundle  $TM$  (Göckeler and Schücker, 1990).

<sup>153</sup> $\mathcal{A}$  may be regarded as a matrix  $\mathcal{A}_{ij}$  of sections of 1-forms in  $\Omega^1$  each taking values in the Lie algebra  $\mathfrak{g}$  of the structure group fiber (stalk)  $\mathbf{G}_x$  of the  $\mathbf{G}$ -bundle (sheaf) in focus. This is essentially Cartan’s definition of connection (Von Westenholz, 1981, Göckeler and Schücker, 1990, Mallios, 1998). In our case, it suffices to define  $\mathcal{A}$  as a (matrix of) section(s) of the  $\mathbf{G}$ -sheaf  $(X, \Omega^1, L^+)$  taking values in the Lie algebra of the structure group  $\mathbf{G} = L^+ = SO(1, 3)^\uparrow$  of orthochronous Lorentz transformations (Baez and Muniain, 1994).

<sup>154</sup>One could also say ‘the Minkowski lightcone opening up in the fiber  $\mathcal{M}_x$  soldered at  $x$ ’.

<sup>155</sup>See previous section and (Raptis, 2000a).

<sup>156</sup>See previous section.

the finsheaf  $\vec{S}_n$  of qausets over the causal set<sup>157</sup>. Also, the adjoint sheaf  $\mathcal{L}_n$ , it too regarded as a local homeomorphism  $\vec{\lambda}^{158}$ , preserves the generator (*ie*, the generating relation or ‘local germ’)  $\vec{\rho}$  of the quantum causal topology of  $\vec{\Omega}_n$ , thus it consists of local, finitary causal and quantal versions of the orthochronous Lorentz group  $L^+ = SO(1, 3)^{\uparrow 159}$ . Altogether, a local  $\vec{\Omega}_n^1$ -section<sup>160</sup> of the  $\mathbf{G}_n$ -finsheaf  $\mathcal{L}_n(\vec{S}_n)$  associates, via the composition  $\vec{\lambda}_n \circ \vec{s}_n$  of the two local homeomorphisms defining the finitary sheaf  $\vec{S}_n$  and its adjoint  $\mathcal{L}_n$ , with a contiguous causal arrow  $x \xrightarrow{*} y$  in the causet  $\vec{F}_n$  a reticular Lorentz local (infinitesimal) transformation in  $[\ell_n^+]_x$ <sup>161</sup> which, in turn, may be thought of as ‘rotating’ the quantal Minkowskian vectors in the stalk  $[\vec{\Omega}_n]_x$ <sup>162</sup> of  $\vec{\Omega}_n(\vec{F}_n)$  over the event  $x$ . Now we see how natural it is to define a finitary spin-Lorentzian connection  $\mathcal{A}_n$  as a local  $\vec{\Omega}_n^1$ -section of the  $\mathbf{G}_n$ -finsheaf  $(\vec{F}_n, \vec{\Omega}_n, D_n, \mathcal{L}_n) =: \vec{\mathcal{M}}_n$ .

However, as we said earlier, since the latter is flat, it admits global sections (Mallios, 1998). Flatness means that  $\mathcal{A}_n \equiv 0$  throughout  $\vec{\mathcal{M}}_n$  (Mallios, 1998), or equivalently, that ‘the connection is identically equal to the trivial constant zero global  $\vec{\Omega}_n^1$ -section of the  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{M}}_n$ ’. In our finitary causal context, we attribute this to the constancy (*ie*, the non-dynamical character) and the transitivity of the inertial Minkowskian causal connection  $\rightarrow$  in  $\vec{F}_n$ , a property that is certainly non-local (Finkelstein, 1988, Raptis, 2000b)<sup>163</sup>. In fact, the ‘unphysicality’ of a chrystalline-rigid causality relation modeled after a transitive, and due to this, global partial order, is already implicitly noted by Zeeman (1964)<sup>164</sup>:

<sup>157</sup>Again, see previous section for relevant definitions.

<sup>158</sup>Also see previous section.

<sup>159</sup>These are the finitary and quantal analogues of infinitesimal causal automorphisms of Minkowski space. As we have said before, the latter, by Zeeman’s work (1964), may be identified with the spin-Lorentz Lie algebra  $\ell^+ = so(1, 3)^{\uparrow} \simeq sl(2, \mathbb{C})$  of  $L^+$ .

<sup>160</sup>The reader should note the arrow over the sub-sheaf space  $\Omega_n^1$  of discrete 1-forms in  $\vec{\mathcal{M}}_n$  which again shows its causal interpretation, as well as its ‘finite resolution index’  $n$ .

<sup>161</sup>Note again the index  $n$  we have given to the local quantum causal symmetries of  $\vec{\Omega}_n$ . In fact, one should write  $I_x$  instead of  $x$  for the index of the stalk of the finsheaf  $\mathbf{G}_n(\vec{S}_n)$ , since the point-event in the qauset  $\vec{\Omega}_n$  corresponding to the event  $x$  of the causal set  $\vec{F}_n$  is the primitive ideal  $I_x$  of the quantum causal incidence algebra  $\vec{\Omega}_n$  associated with the causet  $\vec{F}_n$  (see sections 3 and 4).

<sup>162</sup>See (Raptis, 2000a) for notation and definition of stalks of finsheaves  $S_n(F_n)$ .

<sup>163</sup>See the CEP and its finitary formulation FEP in section 2.

<sup>164</sup>Remark 3 in (Zeeman, 1964). Parenthetically, we may recall from (Zeeman, 1964) that if  $\mathcal{M}$  is Minkowski space and for any two events  $x, y \in \mathcal{M}$   $x < y$  means that  $y - x$  is a future-directed timelike vector (*ie*,  $\|y - x\| := (y - x)^\mu \eta_{\mu\nu} (y - x)^\nu < 0$ ;  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  and  $y_0 > x_0$ ), then the group  $\mathbf{G}$  of all one-to-one maps  $f$  of  $\mathcal{M}$  to itself that preserve ‘ $<$ ’ (*ie*, the ‘causality group’ consisting of the causal automorphisms of  $\mathcal{M}$ ), is isomorphic to the inhomogeneous conformal orthochronous Lorentz group (*ie*, the orthochronous Lorentz group  $L^+$ , together with translations, dilatations and space-reversals, but with spacetime volume-preserving maps being excluded). The latter reflects

The condition for  $f$  to be a causal automorphism is a global condition, but is equivalent (by an elementary compactness argument using the transitivity of  $<$ ) to the following local condition: given  $x \in \mathcal{M}$ , then there is a neighborhood  $N$  of  $x$  such that  $y < z \Leftrightarrow fy < fz, \forall y, z \in N$ . Intuitively this means we need only think of the principle of causality acting in our laboratories for a few seconds, rather than between distant galaxies forever, and still we are able to deduce the Lorentz group...

and explicitly by Finkelstein (1988) who also emphasizes the need for a dynamical local causality:

...the causal relation  $x\mathbf{C}y$ <sup>165</sup> is not local, but may hold for events as far apart as the birth and death of the universe. Since we have committed ourselves to local variables, we abandon  $\mathbf{C}$  for a local causal relation  $\mathbf{c}$ ... We localize the causal relation  $\mathbf{C}$ <sup>166</sup> by taking as basic dynamical variable a relation  $xcy$  expressing *immediate* causal priority<sup>167</sup>.

Indeed, like Finkelstein, we regard the germ relation  $\xrightarrow{*}$  of the local causal topology of  $\vec{F}_n$ , or its finsheaf  $\vec{s}$ -image  $\vec{\rho}$  of  $\vec{\Omega}_n$ , as being dynamically variable. This is achieved by localizing or gauging the finsheaf  $\vec{S}_n$  and its adjoint  $\mathcal{L}_n$ <sup>168</sup> which, in turn, corresponds to implementing a non-zero (non-flat) dynamically variable  $\mathbf{g}_n$ -valued gauge connection  $\mathcal{A}_n$  realized as a local  $\vec{\Omega}_n^1$ -section of the  $\mathbf{G}_n$ -finsheaf  $\vec{\mathcal{P}}_n = (\vec{F}_n, \vec{\Omega}_n, \mathcal{L}_n, \mathcal{D}_n)$ . Thus,  $\mathcal{A}_n$  effectively represents a finitary gravitational dynamics of gausetes.

Since  $\mathcal{A}_n$  represents the dynamics of the germ  $\vec{\rho}$  of the quantum causal topology of the gauset-stalks of  $\vec{\mathcal{P}}_n$ , it is defined locally<sup>169</sup>, thus purely algebraically<sup>170</sup>. The

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the fact that causality, as a partial order, determines the Minkowski metric up to its determinant (spacetime volume form) (Robb, 1914). Note also that in this globally flat Minkowski case Zeeman can form the difference  $y - x$  of Minkowski vectors-a ‘distant comparability’ that is not allowed in the curved case (see discussion above).

<sup>165</sup>Where the  $\mathbf{C}$  in Finkelstein’s paper is the same as our constant transitive partial order causal relation  $\rightarrow$ .

<sup>166</sup>We add ‘ $\mathbf{C}$ ’ at this point of the excerpt for emphasis.

<sup>167</sup>Where the  $\mathbf{c}$  in Finkelstein’s paper is the same as the transitive reduction  $\xrightarrow{*}$  of the partial order causality relation  $\rightarrow$  of the causal set  $\vec{F}_n$  (Raptis, 2000b)-the germ of the causal topology in the finsheaf  $\vec{S}_n$  of gausetes which, by the local homeomorphism  $\vec{s}$  defining it, corresponds to the germ  $\vec{\rho}$  of the quantum causal topology of the gausetes  $[\vec{\Omega}_n]_x$ -the stalks over the coarse point-events of  $\vec{F}_n$ .

<sup>168</sup>Of course, when one localizes or gauges gausetes, it follows that their local quantum causal symmetries are gauged as well.

<sup>169</sup>That is to say, stalk-wise in the sheaf.

<sup>170</sup>Since the stalks are quantally and causally interpreted incidence Rota algebras with the generator  $\vec{\rho}$  of quantum causal topology being defined entirely algebraically with respect to the stalks’ spectra  $\vec{\mathcal{S}}$  as we saw in sections 3 and 4.

detailed algebraic argument that leads to the expression for  $\mathcal{A}_n$  in terms of  $\vec{\rho}$  is left for (Raptis, 2000f). For the present paper it suffices to give the usual gauge theoretic expression for the curvature associated with  $\mathcal{D}_n$ :  $\mathcal{F}_n := \mathcal{D}_n^2 = \mathcal{D}_n \wedge \mathcal{D}_n = [\mathcal{D}_n, \mathcal{D}_n]$ <sup>171</sup> (Göckeler and Schücker, 1990, Baez and Muniain, 1994, Mallios, 1998, Dimakis and Müller-Hoissen, 1999) and note that it is defined entirely locally-algebraically stalk-wise in the sheaf without reference to an ambient geometric base space. As we argued earlier this is quite welcome from the point of view of ‘quantum gravity’<sup>172</sup>.  $\mathcal{F}_n$  may be physically interpreted rather freely in our scheme as a finitary, causal and quantal expression of Lorentzian gravity.

Now that we have mathematically defined and physically interpreted  $\mathcal{A}_n$  (and its curvature  $\mathcal{F}_n$ ) on  $\vec{\mathcal{P}}_n$ , we give an alternative physical interpretation for it more in line with the operational interpretation of finsheaves in (Raptis, 2000a), whereby, the latter were regarded as ‘approximations of the continuous spacetime observables’. So again, let  $X$  be a bounded region in a curved smooth spacetime manifold  $M$  on which  $\mathcal{A}$  lives (in  $\mathcal{P}$ ). As in (Sorkin, 1991) the open sets  $U$  in the locally finite open cover  $\mathcal{U}_n$  of  $X$  were physically interpreted as ‘coarse acts of localization (local determination) of the continuous topology carried by  $X$ ’s point-events’ which, in the finsheaf  $S_n(F_n)$  of (Raptis, 2000a), translates to ‘coarse acts of local determination of the continuous (*ie*,  $C^0$ -topological) spacetime observables’<sup>173</sup>, so similarly we lay out  $\mathcal{U}_n$  to chart roughly the causal topology and the causal symmetries of the bounded spacetime region  $X$  in the gravitational spacetime  $M$ <sup>174</sup>. Then, we organize our observations of the dynamics of local quantum causality into the curved  $\mathbf{G}_n$ -finsheaf of qausets  $\vec{\mathcal{P}}_n$  as described above. In  $\vec{\mathcal{P}}_n$ , we perceive as ‘gravitational gauge potentials  $\mathcal{A}$ ’ the  $\mathbf{g}_n$ -valued  $\vec{\Omega}_n^1$ -sections  $\mathcal{A}_n$ . Now, in the manner that we described the *aufbau* of  $\vec{\mathcal{P}}_n$  in sections 2-5, it is straightforward to interpret  $\mathcal{A}_n$  as ‘equivalence classes of gravitational gauge potentials’ relative to our coarse and dynamically perturbing causal topological observations  $\mathcal{U}_n$ <sup>175</sup>. Equivalently, following verbatim the physical interpre-

<sup>171</sup>Where ‘ $\wedge$ ’ denotes Cartan’s exterior product and ‘ $[\cdot, \cdot]$ ’ ‘commutator’. It follows that  $\mathcal{F}_n$  is a  $\ell_n^+$ -valued section of the  $\vec{\Omega}_n^2$  sub-sheaf of  $\vec{\mathcal{P}}_n$ .

<sup>172</sup>See also Einstein’s quotation at the end of (d) in the next section.

<sup>173</sup>Which by definition preserve the local Euclidean manifold topology of  $X$ .

<sup>174</sup>Following the terminology in Mallios (1998), we call the elements  $U$  of  $\mathcal{U}_n$  ‘coarse (or fat) local gauges’, since they stand for rough acts of measurement of the local (*ie*, the point-event) structure of  $X$ .

<sup>175</sup>The reader must notice here that  $X$  ‘physically exists’ as a background space to the extent that it lends itself to us as a topological substrate on which we may lay (perform, apply, localize) our finitary topological gauges  $\mathcal{U}_n$ -the locally finite pragmatic observations of ‘it’. In this sense it is a ‘surrogate space’ which avails itself to us for performing our quantum discretizations of ‘it’ which, in turn, recover ‘it’ as a local (*ie*, classical smooth) structure at the operationally implausible (ideal),

tation of finsheaves in (Raptis, 2000a),  $\mathcal{A}_n([x])$  stands for a collection of gravitational gauge potentials that are ‘indistinguishable’ at the finite level  $n$  of resolution of  $X$  into its point-events<sup>176</sup>. ‘Indistinguishability’ may be physically interpreted here in a dynamical way as follows: the gravitational field is not perceived as varying between any two events in the same equivalence class  $[x]$ <sup>177</sup>. Furthermore, and here is the operational weight that the sheaf theoretic scheme of ours carries, it is our coarse operations of determination of the dynamical local (quantum) causal topology, which are organized into  $\vec{\mathcal{P}}_n$ , that are effectively encoded in  $\mathcal{A}_n$ , so that, ‘by the end of the day’, it is not the point-events of  $X$  *per se* that carry information about the dynamics of (quantum) causality; rather, it is our own dynamically perturbing observations of them that create ‘it’<sup>178</sup>. Then, the General Relativistic character of our sheaf theoretic scheme may be summarized in the following: the finitary gravitational connection  $\mathcal{D}_n$  on  $\vec{\mathcal{P}}_n$  is a sheaf morphism (Mallios, 1998), which means that the dynamics of local quantum causality is  $\mathcal{U}_n$ -independent<sup>179</sup>. This is the (fin)sheaf theoretic version of the principle of General Covariance of GR with a strong local-quantal flavor<sup>180</sup>.

## 6. DISCUSSION OF THE SOUNDNESS OF $\vec{\mathcal{P}}_n$ AND OTHER RELEVANT ISSUES

In this last section we present four arguments that support that  $\vec{\mathcal{P}}_n$  is a sound model of a finitary, causal and quantal version of Lorentzian gravity.

(a) The FEP of section 2 is satisfied in  $\vec{\mathcal{P}}_n$ , since the latter’s stalks  $\vec{\Omega}_n$  are local isomorphs of a locally finite, causal and quantal version of Minkowski space (sections 3 and 4).

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classical limit of observations of infinite power (or energy) of resolution (Raptis and Zapatin, 2000, Raptis, 2000a). See also remarks in (c) of the next section.

<sup>176</sup>Thus, we tacitly assume that the spacetime events are not only surrogate carriers of  $X$ ’s physical topology (Sorkin, 1991, Raptis, 2000a), but also of its other physically observable fields (attributes)-the gravitational field being the one in focus here.

<sup>177</sup>This interpretation is consistent with our FEP of section 2 which, in effect, held that in a finsheaf over a causet  $\vec{F}_n$  obtained by  $\mathcal{U}_n$  as described in sections 3 and 4, all the frames in the ‘fat (coarse) stalks’  $\vec{\Omega}_n([x])$  over  $[x]$  are inertial relative to each other.

<sup>178</sup>See the active operational interpretation of dynamical local quantum causality at the end of section 4. See also (c) in the next section.

<sup>179</sup>For a short discussion of this apparently paradoxical situation, namely, that our own coarse local observations  $\mathcal{U}_n$  create the dynamical local quantum causality whose dynamics is subsequently expressed in a  $\mathcal{U}_n$ -invariant (*ie*, gauge independent) way, see (c) in the next section.

<sup>180</sup>Since  $\mathcal{D}_n$  respects the linear quantum kinematical structure (*ie*, the coherent quantum superpositions of quaset) stalk-wise in  $\vec{\mathcal{P}}_n$ . See also (b’) in the next section.



The FLRP of section 2 holds in  $\vec{\mathcal{P}}_n$ , since the latter's group  $\mathbf{G}_n$ -stalks are finitary, causal and quantal versions of the orthochronous Lorentz structure group  $L^+$  of local causal symmetries of GR.

The FLP of section 2 is satisfied in  $\vec{\mathcal{P}}_n$ , since the latter's gauset stalks are sound models of local quantum causality (section 3 and (Raptis, 2000b)).

The FLSP of section 2 holds in  $\vec{\mathcal{P}}_n$ , since the gausets residing in its stalks coherently superpose with each other (sections 3, 4 and (Raptis, 2000b)).

In section 2 we posited that a sound mathematical model of a finitary curved quantum causal space should meet structurally these four 'physical axioms'. Indeed, the structure of  $\vec{\mathcal{P}}_n$  does meet them.

(b)  $\vec{\mathcal{P}}_n$  is an algebraic model for finitary, local and dynamically variable quantum causality which inherits its operational meaning from the pragmatic and quantal interpretation given to quantum topological incidence algebras in (Raptis and Zapatrin, 2000), hence also to their causal relatives in (Raptis, 2000b). Moreover, from (Raptis and Zapatrin, 2000) it inherits its essentially alocal character, while, together with the physical interpretation of finsheaves (of algebras) in (Raptis, 2000a), it manifests its essential non-commitment to spacetime as a background geometrical  $C^\infty$ -smooth manifold.

(c) The local structure of classical gravitational spacetime, namely, the event and the space of Minkowskian directions tangent to it, arise only at the operationally ideal limit of infinite localization<sup>181</sup> of an inverse system of  $\vec{\mathcal{P}}_n$ s (Raptis, 2000a). The latter limit, yielding the classical gravitational sheaf  $\mathcal{P}$  in a manner analogous to how the sheaf  $S(X)$  of continuous functions on a topological spacetime manifold arises at infinite refinement of topological finsheaves  $S_n$  similar to our causal  $\vec{\mathcal{P}}_n$ s (Raptis, 2000a), may be physically interpreted as Bohr's correspondence principle (Raptis and Zapatrin, 2000). This further supports the quantal character of  $\vec{\mathcal{P}}_n$ . All in all, putting together the physical interpretations of the theoretical schemes proposed in (Raptis and Zapatrin, 2000), (Raptis, 2000b) and (Raptis, 2000a) that are amalgamated into our model  $\vec{\mathcal{P}}_n$  for the dynamics of finitary quantum causality as described in sections 3-5, we may summarize the physical interpretation of  $\vec{\mathcal{P}}_n$  to the following: it represents alocal, discrete, causal and quantal operations of determination of the dynamics of causality and its symmetries in a bounded region of a curved smooth

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<sup>181</sup>That is to say, at the operationally ideal situation of employment of an infinite power (or energy) to resolve spacetime into its point-events. As we explained in section 3, this is theoretically implausible too due to the fundamental conflict of the principles of Equivalence and Uncertainty on which gravity and the quantum are founded at energies (*ie*, microscopic powers of resolution) higher than  $\hbar t_P^{-1}$ .

spacetime manifold with the latter not existing in a physically significant sense<sup>182</sup>, but only viewed as providing a surrogate scaffolding on which we base (*ie*, locally solder<sup>183</sup>) our own operations of observing ‘it’, which are then suitably organized into algebra finsheaves. This collective physical interpretation of  $\vec{\mathcal{P}}_n$  is well in accord with the general philosophy of quantum theory holding that inert, background, geometrical ‘state spaces’ and their structures, such as spacetime and its causal structure, ‘dissolve away’, so that what remains and is of physical significance, the ‘physically real’ so to speak, is (the algebraic mechanism<sup>184</sup> of) our own actions of observing ‘it’ (Finkelstein, 1996).

In section 4 we stretched even further this ‘observer-dependent physical reality’ essence of quantum theory to an ‘observation-created physical causality’ with the introduction of the ‘quantum causal potential relative to our coarse observations’ observable which was subsequently seen to be the dynamically variable entity represented by the finitary connection  $\mathcal{A}_n$  on  $\vec{\mathcal{P}}_n$  (section 5) only to find that a (fin)sheaf theoretic version of the principle of General Covariance of GR holds in our model, namely, that dynamics is formulated in terms of equations involving the connection  $\mathcal{D}_n$  which is the main finsheaf morphism in  $\vec{\mathcal{P}}_n$  (Mallios, 1998). Thus, our mathematical expressions of ‘physical laws’ are not observation-dependent<sup>185</sup>. This points to the following seemingly paradoxical interpretation of our scheme: the observer acts as a ‘law-maker’ when she observes<sup>186</sup> and as a ‘law-seeker’ when she communicates her observations<sup>187</sup>. There is no conflict, for as Finkelstein (1996) notes:

...Since we and our medium are actually a quantum entity, the goal of knowing the dynamical law completely seems to be a typically ontic<sup>188</sup> one. This completeness

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<sup>182</sup>Not being ‘physically real’, so to speak.

<sup>183</sup>That is to say, we restrict our experimental activity in laboratories of finite spatio-temporal extent. This is the operational analogue of Zeeman’s remark in the quotation from (Zeeman, 1964) given in the previous section that “*we need only think of the principle of causality acting in our laboratories for a few seconds, rather than between distant galaxies forever*” although the time scale of observations of quantum causality is expected to be of the order of Planck time ( $\approx 10^{-42}s$ ), not of few seconds.

<sup>184</sup>Or ‘structure’.

<sup>185</sup>That is to say, physical laws are  $\mathcal{U}_n$ -gauge independent or invariant under our coarse and dynamically perturbing local measurements of ‘spacetime’ (Mallios, 1998).

<sup>186</sup>By establishing causal connections between the events that she observes.

<sup>187</sup>That is to say, when she ‘objectifies’ her actions of determination of ‘it’ to others by organizing the coarse causal nexus she has perceived in ‘it’ to structures (sheaves) so that the dynamics  $\mathcal{D}_n$  of this causal nexus (topology) is independent of her ‘subjective’ coarse measurements of ‘it all’ in  $\mathcal{U}_n$ .

<sup>188</sup>In (Finkelstein, 1996) ‘ontic’ refers to the classical existential ideal due to Plato which holds that ‘objects exist independently of our own modes of perceiving them’. In physics, this Platonic

must prove as counterphysical as the others we have already encountered. Law-seekers are in some part law-makers as well. Just as we influence the laws of geometry slightly by moving masses, we may influence any law of dynamics at least slightly by our own actions...

After all, such an apparently conflicting duality may be necessary for a genuine synthesis of the quantum with relativity (Finkelstein, 1996)-a synthesis which appears to be at the heart of the problem of ‘quantum gravity’ *per se*.

(d) Causet theory (Bombelli *et al.*, 1987, Sorkin, 1990) addresses the problem of ‘quantum gravity’ in locally finite, causal and, to some extent, quantal terms<sup>189</sup> from a non-operational point of view (Sorkin, 1995). Finkelstein’s Quantum Net Dynamics (1988, 1989, 1991) and its subsequent generalization, Quantum Relativity Theory (1996), address the same problem in almost the same terms, but from an ‘entirely operational’<sup>190</sup> point of view. Our finsheaf theoretic model for finitary Lorentzian quantum gravity brings together Finkelstein’s and Sorkin’s approaches under a ‘purely algebraic roof’ and to some extent vindicates their fundamental insight that the problem of ‘quantum gravity’ may be solved or, at least, be better understood, if it is formulated as ‘the dynamics of an atomistic local quantum causal topology’. At least, it certainly goes some way to vindicate Einstein’s hunch: “*Perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is to the elimination of continuous functions from physics*” (Einstein, 1936), and it complies with his more general and imperative intuition (Einstein, 1956) that:

One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality. But nobody knows how to obtain the basis of such a theory.

We conclude the present paper by briefly discussing six physico-mathematical issues that derive from our finsheaf model of finitary Lorentzian quantum gravity:

(a’) The first issue is about localization. The duality between topological Rota incidence algebras and poset finitary substitutes has been established (Raptis and

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ideal translates to: ‘physical objects exist independently of our own actions of determination of their physical properties’.

<sup>189</sup>For instance, a quantum dynamics for causets is sought after a covariant path integral or ‘sum over causet histories’ scenario (Sorkin, 1990).

<sup>190</sup>That is to say, ‘pragmatic’.

Zapatrin, 2000). Similarly, the physical duality between spacetime observables and spacetime states (point-events) was discussed in (Raptis, 2000a). Categorically speaking, this duality corresponds to a contravariant functor from the category of poset-finitary substitutes/poset-morphisms to the category of topological incidence algebras/algebra-homomorphisms (Raptis and Zapatrin, 2000, Zapatrin, 2000)<sup>191</sup>. Localizations in the first category are represented by inverse limits as in (Sorkin, 1991)<sup>192</sup>, while in the second, by direct limits<sup>193</sup>. Inverse and direct limits are formal (categorical) processes of topological localization dual (opposite) to each other (Dodson, 1988). In the present paper, where we also dealt with operations of localization of the main causal-topological observable  $\mathcal{A}_n$  in a curved finitary causal space, the finsheaf approach with its ‘coarse direct limit localizations’ as in (Raptis, 2000a) seems rather natural to adopt for qauset-localizations (to ‘coarse or fat stalks’) and for the coarse gravitational gauge potential  $\mathcal{A}_n$  that twists them ‘there’.

(b’) Related to our ‘localization’ comments above, is that in a finitary curved quantum causal space similar to ours, Finkelstein’s causal net (1988), gravity may be thought of as arising from breaking quantum coherence in the net—a phenomenon which gives rise to vectors in flat Minkowski space (Selesnick, 1994). Since the latter constitute the local structure of a curved spacetime by the CEP, it seems that the very acts of event-localization are responsible for this lifting of ‘quantum coherence’, thus rendering the FLSP of section 2 ineffective. Hence, something like (classical) spin-Lorentzian gravity results when one attempts to localize or gauge Finkelstein’s quantum spacetime net by breaking its quantum coherence (Selesnick, 1994). Similarly in our scheme,  $\mathcal{A}_n$ , which is the result of localizing or gauging quantal Minkowskian qausets in the finsheaf  $\vec{\mathcal{P}}$ , may be thought of as somehow breaking the local (kinematical) quantum superpositions of qausets stalk-wise, by establishing incoherent ‘distant linear combineability of qausets’<sup>194</sup>. On the other hand, the metric  $\mathcal{A}_n$ , by taking values in the local Rota algebra homomorphisms of the qauset-stalks, respects their local linear superpositions stalk-wise. There is no contradiction if we assume that at finite and still quantal level of resolution of  $\vec{\mathcal{P}}_n$  the gravitational connection still preserves

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<sup>191</sup>This functorial equivalence between topological posets and their incidence algebras is used in a most fruitful way in (Raptis, 2000e) to define a ‘quantum topos’ of scheme theoretic localizations of qausets (see  $f'$  below).

<sup>192</sup>Such inverse limits yield the finest or ‘smallest’ open sets containing the spacetime point-events in  $X$  (Sorkin, 1991).

<sup>193</sup>Such direct limits yield the finest fiber spaces of continuous functions over  $X$ ’s point events  $x$ —the stalks of the sheaf in focus consisting of germs of continuous functions at  $x$  for every  $x$  in  $X$  (Raptis, 2000a).

<sup>194</sup>That is to say, it allows for superpositions of qausets that live in different stalks in  $\vec{\mathcal{P}}$ . See section 5.

local coherent quantum suppositions of qausets<sup>195</sup> and only at the ideal classical limit of infinite point-event localization coherence is broken and classical gravity  $\mathcal{A}$  on  $\mathcal{P}$  emerges as in (Selesnick, 1994)<sup>196</sup>.

Furthermore, Finkelstein (1988) has entertained the idea that such quantum coherence-breaking localization processes are in fact physical quantum condensations of the net<sup>197</sup> and classical spacetime is thought of as arising from such ‘phase transitions’ that the net undergoes. It would certainly be worthwhile to try and relate this theory to our curved qausets and their decoherence to classical gravitational spacetime.

(c’) The third issue is about spacetime dimensionality. As we mentioned in the introduction, we did not consider questions of dimensionality in the present paper. However, since dimensionality is a topological invariant and our finsheaf scheme is supposed to be an account of (at least the kinematics of) a dynamical local quantum causal topology, it should somehow address the problem of the dimensionality of spacetime and its dynamical fluctuations at quantum scales. Bombelli *et al.* (1987) and Sorkin (1990), for instance, entertain the idea that spacetime dimensionality is a statistical variable varying with the power of ‘fractally’ refining or resolving causet. On the other hand, as it was also mentioned at the end of (Raptis and Zapatrin, 2000), albeit in a quantum topological not a causal context like ours, our qausets should not be thought of as being of statistical nature so that the classical spacetime dimensionality, as well as the other classical spacetime properties such as the differential and local Lorentz-causal structures arise at some ‘thermodynamic limit’ as if they are ‘sta-

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<sup>195</sup>In a sense, by locally preserving the quantum coherence of qausets (*ie*, stalk-wise),  $\mathcal{A}_n$  qualifies for a quantal sort of gravitational action. It follows that, since the qauset ‘paths’ (Raptis and Zapatrin, 2000, Raptis, 2000*b*) coherently interfere, so does the gravitational connection  $\mathcal{A}_n$  that is defined on them. Thus, some kind of reticular (*ie*, ‘innately or inherently regularized’) path integral quantization of gravity, by considering for instance a local spin-Lorentz covariant path integral quantization scenario over the space of qauset connections  $\mathcal{A}_n$  on the principal finsheaves thereof, is already implicit in our scheme. However, the ambitious project to explicate this and put it on a rigorous footing must be left for a future investigation.

<sup>196</sup>It is interesting to mention at this point that Selesnick ascribes the emergence of classical gravity on Finkelstein’s quantum net by localizing or gauging the  $SL(2, \mathbb{C})$  symmetries of the quantum relativistic binary alternative to “*what a macroscopic observer having sufficiently limited powers of spacetime resolution perceives as this gauging*”. Our finitary model  $\tilde{\mathcal{P}}_n$  for Lorentzian gravity also initially derives from such pragmatic coarse observations of the topological structure of spacetime as in (Sorkin, 1991, Raptis, *a*), but in the manner of (Raptis, 2000*b*) it also acquires its essentially causal and quantal interpretation, thus unlike Selesnick’s account, its gravity should be placed at the quantum and not the classical side of Heisenberg’s *schnitt*. Only at the ideal limit of infinite spacetime localization does classical gravity emerge (on  $\mathcal{P}$ ).

<sup>197</sup>Hence the epithet ‘superconducting’ given to such quantum causal nets (Finkelstein, 1988).

tistical average attributes of spacetime'. Rather, since classical spacetime is expected to arise at the classical decoherence limit from curved finsheaves of qausets in a way similar to Finkelstein's net condensation scenario, we too anticipate that spacetime dimensionality is a long range order parameter of the condensate (Finkelstein, 1988, 1996)<sup>198</sup>.

Furthermore, Finkelstein (1996) anticipates that not only the spacetime dimensionality but also that the spacetime metric  $g_{\mu\nu}$ , the basic dynamical variable in GR, is such a long range parameter of a quantum condensate, so that the requirement for the connection to be metric<sup>199</sup> is only for our theoretical convenience and not a truly fundamental aspect of Nature. Our theoretical model  $\vec{\mathcal{P}}_n$  of finitary Lorentzian quantum gravity also invites the assumption of a non-distorting connection  $\mathcal{A}_n$ , thus the latter is 'model-dependent'<sup>200</sup> and may prove to be physically not fundamental.

(*d'*) The fourth issue is about 'space-likeness versus time-likeness'. At the end of (Sorkin, 1991), in a comparison between topological posets and causets, it is explicitly mentioned that the former, say  $P$ , may be used to represent the coarse spatial topology of a 'thickened'<sup>201</sup> space-like Cauchy hypersurface in a globally hyperbolic spacetime manifold  $M$  which approximates a fundamental causet  $C$ <sup>202</sup>. Our qausets on the other hand are quantum causal replacements of the continuous spacetime 4-manifold and are essentially of time-like nature; they are not topological approximations-proper to spatial 3-submanifolds of  $M$ . The principle of classical spacetime causality, which is modeled by assuming that  $M$  is globally hyperbolic, may also be a long range (global) spacetime property arising from the decoherence of an ensemble of fundamental alocal causet substrata.

(*e'*) The fifth issue we would like to address here concerns gravity as a Yang-Mills type of gauge theory. In the Einstein-Cartan theory (Göckeler and Schücker, 1990) or in Ashtekar's self-dual formulation (Ashtekar, 1986, Baez and Muniain, 1994) the gravitational spin-Lorentzian gauge potential  $\mathcal{A}$ <sup>203</sup> tells only half of the gravitational

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<sup>198</sup>Thus, from a causet point of view, spacetime dimensionality is a quantum variable, but from a causet point of view it is a classical random variable, and there is a significant difference between these two kinds of dynamical variables (David Finkelstein in private correspondence). We may attribute this distinction to the possibility of coherent quantum superpositions between qausets and the absence of such a possibility for causets (Raptis, 2000*b*).

<sup>199</sup>That is to say, that there is no spacetime distortion (Finkelstein, 1996).

<sup>200</sup>That is to say, 'theoretically convenient'. See previous section.

<sup>201</sup>We equivalently use the epithet 'fat' for 'coarse' approximations (see above).

<sup>202</sup>Notice here that in the same way that we criticized the term 'approximations' when applied to describe qausets in the introduction, Sorkin also thinks of the classical spacetime manifold  $M$  as an approximation to causets, not the other way around.

<sup>203</sup>In the Einstein-Cartan theory it is customary to write  $\omega$  for the spin connection instead of  $\mathcal{A}$ .  $\omega$

story in that there is also the (complex) frame field  $e_a^I$ <sup>204</sup> gravitational variable. ‘ $I$ ’ is an internal (spin) index, while ‘ $a$ ’ an external spacetime index. Put together into a Palatini-like action functional and varied independently of each other the two gravitational variables  $\mathcal{A}$  and  $e_a^I$  yield two sets of equations: variation with respect to  $\mathcal{A}$  yields an equation between it and the usual Christoffel metric connection  $\Gamma$  of GR on  $M$  (in  $\mathcal{P}$ ) which establishes the equivalence between the original metric formulation of GR in terms of the Levi-Civita  $\Gamma$  and the gauge theoretic one in terms of the spin-Lorentzian  $\mathcal{A}$ , while variation with respect to  $e_a^I$  yields the usual vacuum Einstein equations (Baez and Muniain, 1994). In our finsheaf of qausets scheme we have not explicitly given finitary, causal and quantal gravitational variables corresponding to the *vierbein*. We only elaborated on  $\mathcal{A}_n$  on  $\vec{\mathcal{P}}_n$ . It is implicit however that, since the dynamically variable qauset-stalks of  $\vec{\mathcal{P}}_n$  locally define co-vector spaces of the form  $\tilde{\Omega}_n^1$ , cross-sections of their duals correspond to reticular, causal and quantal versions of the *vierbein* observables<sup>205</sup>. Hence, as we also made it clear in the introduction, it is perhaps more accurate to interpret our gauged finsheaf of qausets theoretic scheme  $\vec{\mathcal{P}}_n$  as a finitary, causal and quantal replacement of the model  $\mathcal{P}$  of curved spacetime structure in which classical Lorentzian gravity is formulated as a gauge theory, rather than directly as a finitary, causal and quantal substitute of GR *per se*. For instance, we have not even given the finitary, causal and quantal analogues of the Einstein equations in  $\vec{\mathcal{P}}_n$ . We do this in a forthcoming paper (Raptis, 2000f). Thus, it is perhaps more appropriate to view  $\vec{\mathcal{P}}_n$  as a kinematical structure that supports the (germ of the) dynamics which is represented by the non-trivial  $\mathcal{D}_n$  on it. To lay down the kinematics before the dynamics for qausets may be viewed as the first essential step in the development of the theory. For example, Sorkin (1995), commenting on the influence that Taketani’s writings (1971) have exerted on the construction of causet theory and its application to quantum gravity, stresses that “...*there is nothing wrong with taking a long time to understand a structure ‘kinematically’ before you have a real handle on its dynamics*”. In our case, there should be no confusion whatsoever when one interprets  $\vec{\mathcal{P}}_n$  directly as some kind of ‘finitary and causal Lorentzian quantum gravity’ provided one remembers the remarks above.

( $f'$ ) Finally, Selesnick (1991), working on a correspondence principle for Finkelstein’s quantum causal net in (1988), speculated on a topos theoretic formalization of the net, where relativity and the quantum will harmoniously coexist, and one that is more fundamental than  $\mathbf{Sh}(X)$ -the category of sheaves of sets over the topological

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is also what Selesnick obtains from gauging Finkelstein’s net as briefly alluded to above (Selesnick, 1994). Ashtekar uses only the self-dual part  $\mathcal{A}^+$  of the gravitational gauge potential.

<sup>204</sup>Also known as ‘co-moving tetrad’ or ‘*vierbein*’.

<sup>205</sup>4-dimensionality and  $\mathbb{C}$ -coefficients arguments aside, as we explained earlier.

spacetime manifold  $X$  in which both classical and quantum field theories are currently formulated. Indeed, a (quantum ?) topos organization of finsheaves of gausets may prove to be the structure Selesnick anticipated. Searches for such a fundamental ‘quantum topos’ in which ‘quantum gravity’ may be formulated rather naturally have already been conducted from a slightly different point of view than gauset theory proper (Raptis, 1998, 2000c)<sup>206</sup>. In any case, whatever this elusive ‘quantum topos’ structure turns out to be, it will certainly shed more light on a very important mathematical question that has occupied mathematicians for some time now and which can be cast as the following puzzling analogy:  $\frac{\text{locales}}{\text{quantaes}} = \frac{\text{topoi}}{?}$ <sup>207</sup>.

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<sup>206</sup>In a nutshell, it has been shown that the classical (*ie*, Boolean) topos of quaternion algebras  $\mathbb{H}$  corresponds to classical Minkowski spacetime  $\mathcal{M}$  (Trifonov, 1995). In (Raptis, 1998, 2000c) it has been argued that it is precisely the (global) associativity of the quaternion product that it is responsible for the flatness of  $\mathcal{M}$ . In section 3 we too entertained the idea that the (global) flatness of causets or their corresponding gausets, which determine classical and quantal versions of  $\mathcal{M}$ , is due to their transitivity or their associativity, respectively. In (Raptis, 1998, 2000c) a non-classical (*ie*, non-Boolean or quantum ?) topos of non-associative algebras is proposed to model a finitary curved quantum causal space similar to the gauged finsheaves of gausets that we proposed in the present paper or their respective finschemes and their quantum topos organization in (Raptis, 2000e). The latter may be regarded as a straightforward attempt at arriving at ‘the true quantum topos for quantum gravity (and quantum logic)’ via gausets and their non-commutative schematic localizations.

<sup>207</sup>Jim Lambek and Steve Selesnick in private correspondence. Briefly, a topos or a locale like, say,  $\mathbf{Sh}(X)$ , may be thought of as ‘a generalized pointless topological space’ and it may be interpreted as a universe of variable sets (Selesnick, 1991, Mac Lane and Moerdijk, 1992). Such an interpretation is very welcome from the physical point of view that we have adopted in the present paper according to which we are looking for a mathematical structure to model the dynamical variations of gausets in a way that does not commit itself to spacetime as an inert background geometrical point-event manifold  $X$ . However, much more work has to be done to put finsheaves of gausets on a firm topos theoretic basis and study the resulting structure’s quantal features. As we said, significant progress in this direction is analytically presented in (Raptis, 2000e).



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